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Linear-quadratic optimal control for time-varying descriptor systems via space decompositions

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Abstract

This paper aims at solving the linear-quadratic optimal control problems (LQOCP) for time-varying descriptor systems in a real Hilbert space. By using the Moore-Penrose inverse theory and space decomposition technique, the descriptor system can be rewritten as a new differential-algebraic equation (DAE), and then some novel sufficient conditions for the solvability of LQOCP are obtained. Especially, the methods proposed in this work are simpler and easier to verify and compute, and can solve LQOCP without the range inclusion condition. In addition, some numerical examples are shown to verify the results obtained.

Keywords linear-quadratic optimal control problem (LQOCP), time-varying descriptor system, Moore-Penrose inverse, space decomposition

1 Introduction

For the DAE,

$$(\boldsymbol{A}(t)\boldsymbol{x}(t))' = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t); \quad t \in [0, T] \tag{1}$$

where $T \geqslant 0$, $\boldsymbol{A}(t)$, $\boldsymbol{B}(t)$, $\boldsymbol{C}(t)$ are continuous operator-valued functions, $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ are continuous functions in time t, if $\boldsymbol{A}(t)$ is singular, the DAE can be called a time-varying descriptor or singular system. The descriptor system is a kind of system with more general formulation and can be used to describe many real system models[1] more conveniently and naturally, which has been widely concerned in recent years and applied in chemical engineering, management science, aerospace engineering, robotic technology and so on. Rosenbrock[2] first proposed the concept of descriptor systems to study some structural properties of a class of dynamical systems. In 1989, Dai[3] published a classic monograph on the basic theory of descriptor systems, after that, the descriptor system theory has been developed rapidly, including the solvability of initial value problems, multi-point boundary value problems (BVPs), the stability and robust stability of solutions[4-8]. For instance, the problems of robust control and filtering for descriptor systems were investigated in Ref. [4]. In Ref. [5], the generalized Lyapunov inequality was applied to describe the dissipative Hamiltonian equations, and some sufficient conditions for the stability of the port-Hamiltonian descriptor system were given. Terasaki et al. [6] considered the minimum controllability problem on linear structural descriptor systems by means of

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Dulmage-Mendelsohn decomposition. In Ref. [7], the authors considered the singular optimal control problem (OCP) of minimizing the energy supply of linear dissipative port-Hamiltonian descriptor systems. A sliding mode observer design method was proposed in Ref. [8] to estimate the states and unknown inputs of a class of non-infinite observable descriptor systems.

The OCP is to find an optimal control method from all possible control schemes so that the given performance index attains the optimal value for a controlled dynamic system. An OCP is said to be an LQOCP, if the control schemes and performance index can be determined by linear and quadratic functions of state components and control variables, respectively. In recent years, the research on the optimal control theory has made an increasing progress[9-22]. Sun et al.[9] focused on the stochastic LQOCP in an infinite horizon with constant coefficients. Based on Riccati equations and strongly continuous quasi semi-groups, LQOCPs for non-autonomous linear control systems were considered in Ref. [10]. Besides, Ref. [11] considered the stochastic LQOCP and revealed the deep characteristics of the stochastic turnpike problem. In Ref. [12], the authors used Bellman's dynamic programming principle to propose a reinforcement learning method and solved the infinite horizon continuous-time stochastic linear quadratic problem. LQOCP for stochastic systems with partial information was studied in Ref. [13].

The OCP for different linear systems has been concerned by many researchers. Kurina[16] studied the discrete OCP in descriptor systems by using the projection method, and the time-varying case of version was solved in Ref. [17]. After that, based on the method of solving DAEs initial value problems, an algorithm for solving LQOCP of two-steps descriptor systems was proposed in Ref. [18], and Kurina et al.[19] studied the LQOCP where the performance index contained two small parameters of different orders. More recently, OCP for discrete and continuous stochastic descriptor systems, indefinite LQOCP for rectangular descriptor systems and linear descriptor Markov jump systems were studied[20-22].

It is worth noting that the space decomposition is a useful method to get the spectrum of operator completion problems[23], the quadratic numerical range of operators[24], the invertibility of operator matrices[25], etc. Inspired by Ref. [17], this paper aims at solving LQOCPs for time-varying descriptor systems, and gets some new sufficient conditions for solvability of the associated linear BVP based on space decompositions. Different from most of the existing literatures, (e. g. Refs. [17-22]), a new technique combining the Moore-Penrose inverse with space decomposition, is proposed in this paper to deal with the LQOCP for time-varying descriptor systems, which is more effective to study the non-singular operator-valued function for descriptor systems, and can relax some restriction conditions, such as removing the range inclusion condition in Ref. [17]. This method can greatly reduce the difficulty of analysis and decrease the burden of calculations, and the obtained results in this paper can be applied into some real systems to solve the OCP. The main contributions are as follows.

1) By means of the Moore-Penrose inverse and space decomposition, the time-varying descriptor system can be reformulated as a new DAE.

2) This paper uses the space decomposition method to obtain a new sufficient condition for the control optimality. By comparing with the result in Ref. [17], this sufficient condition does not need to define other variables except the allowable control input and the corresponding solution.

3) The solvable conditions of the linear BVP are given by the invertibility or positive semi-definiteness of some suitable operators, which cannot be necessary to judge the invertibility of associated operator matrix and the range inclusion condition, in contrast to Ref. [17] using non-negative Hamiltonian systems.

The rest of this paper is arranged as follows. In Sect. 2, some preliminaries are introduced. Sect. 3 presents a sufficient condition for the control optimality. Sect. 4 discusses the solvability of the LQOCP. Examples are given in Sect. 5. Finally, the conclusion is summarized in Sect. 6.

2 Preliminaries

Some useful definitions and symbols for linear operators in real Hilbert spaces are introduced.

Let X, Y, Z be real Hilbert spaces and $L(X,Y)$ be the collection of all bounded linear operators from X to Y. For notational simplicity, we denote $L(X):=L(X,X)$. If the operator $\boldsymbol{N}\in L(X,Y)$ has closed range, the Hilbert spaces X and Y admit the orthogonal decomposition $X=K(\boldsymbol{N})^{\perp}\oplus K(\boldsymbol{N})$ and $Y=I(\boldsymbol{N})\oplus I(\boldsymbol{N})^{\perp}$, where $K(\cdot)$ and $I(\cdot)$ are the kernel and image of a linear operator; $\perp$ and $\oplus$ denote the orthogonal complement and orthogonal sum operations, respectively. Based on such space decompositions, an operator in $L(X,Y)$ would be represented by a matrix form with some special characteristics[25]. Define $\boldsymbol{P}$ and $\boldsymbol{Q}$ as the orthogonal projections onto $K(\boldsymbol{N})$ along $K(\boldsymbol{N})^{\perp}$ and onto $I(\boldsymbol{N})^{\perp}$ along $I(\boldsymbol{N})$, respectively. It is obvious that the projections $\boldsymbol{P}$ and $\boldsymbol{Q}$ exist uniquely and $I(\boldsymbol{P})=K(\boldsymbol{N})$, $I(\boldsymbol{Q})=I(\boldsymbol{N})^{\perp}$.

Definition 1[26] Given $\boldsymbol{N}\in L(X,Y)$, if there is an $\boldsymbol{M}\in L(Y,X)$ such that $\boldsymbol{NMN}=\boldsymbol{N}$, $\boldsymbol{MNM}=\boldsymbol{M}$, $(\boldsymbol{NM})^{*}=\boldsymbol{NM}$, $(\boldsymbol{MN})^{*}=\boldsymbol{MN}$, then $\boldsymbol{M}$ is unique, called as the Moore-Penrose inverse of $\boldsymbol{N}$, where $*$ is the adjoint operation.

As usual, $\boldsymbol{N}^{+}:=\boldsymbol{M}$ and, in this case, $\boldsymbol{M}$ is said to be Moore-Penrose invertible, where $+$ is the Moore-Penrose inverse operation.

As well-known, $\boldsymbol{NN}^{+}=\boldsymbol{I}-\boldsymbol{Q}, \boldsymbol{N}^{+}\boldsymbol{N}=\boldsymbol{I}-\boldsymbol{P}$, and $\boldsymbol{N}$ is Moore-Penrose invertible if and only if $\boldsymbol{N}$ has closed range, which is usually defined as the normal solvability of $\boldsymbol{N}$, where $\boldsymbol{I}$ is the identity matrix.

3 A sufficient condition for control optimality

In this section, a new DAE of the time-varying descriptor system and a novel sufficient condition for control optimality by means of the Moore-Penrose inverse and space decomposition are obtained.

Consider the following descriptor system

$$(\boldsymbol{A}(t)\boldsymbol{x}(t))'=\boldsymbol{C}(t)\boldsymbol{x}(t)+\boldsymbol{B}(t)\boldsymbol{u}(t);\quad t\in[0,T] \tag{2}$$

$$\boldsymbol{A}(0)\boldsymbol{x}(0)=\boldsymbol{y}_0 \tag{3}$$

and minimize the associated quadratic cost functional

$$J(\boldsymbol{u}(t),\boldsymbol{x}(t))=\frac{1}{2}\langle\boldsymbol{x}(T),\boldsymbol{V}\boldsymbol{x}(T)\rangle+\frac{1}{2}\int_0^T(\langle\boldsymbol{x}(t),\boldsymbol{W}(t)\boldsymbol{x}(t)\rangle+2\langle\boldsymbol{x}(t),\boldsymbol{S}(t)\boldsymbol{u}(t)\rangle+\langle\boldsymbol{u}(t),\boldsymbol{R}(t)\boldsymbol{u}(t)\rangle)\mathrm{d}t \tag{4}$$

Here, $T\geqslant 0$ and $\boldsymbol{y}_0\in Y$ are fixed, $\boldsymbol{x}(t)\in X$ and $\boldsymbol{u}(t)\in Z$, the operator-valued function $\boldsymbol{A}(t)\in L(X,Y)$ is continuously differentiable and Moore-Penrose invertible for all $t\in[0,T]$. $\boldsymbol{C}(t)\in L(X,Y)$, $\boldsymbol{B}(t)\in L(Z,Y)$ and $\boldsymbol{S}(t)\in L(Z,X)$ are all continuous. $\boldsymbol{W}(t)\in L(X)$ and $\boldsymbol{R}(t)\in L(Z)$ are continuous and self-adjoint for all $t\in[0,T]$, $\boldsymbol{V}\in L(X)$ is self-adjoint and independent of t. The symbol "$'$" stands for $\mathrm{d}/\mathrm{d}t$.

Note that the solution $\boldsymbol{x}(t)$ is continuous such that $\boldsymbol{A}(t)\boldsymbol{x}(t)$ is continuous differentiable. An admissible control input $\boldsymbol{u}(t)$ of Eqs. (2) – (3) is continuous, under which Eqs. (2) – (3) have a solution.

Since $\boldsymbol{A}(t)$ is Moore-Penrose invertible for any $t\in[0,T]$, it follows that

$$X=K(\boldsymbol{A}(t))^{\perp}\oplus K(\boldsymbol{A}(t)) \tag{5}$$

$$Y=I(\boldsymbol{A}(t))\oplus I(\boldsymbol{A}(t))^{\perp} \tag{6}$$

Then for any $t\in[0,T]$, the orthogonal projections $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ can be determined by $I(\boldsymbol{P}(t))=K(\boldsymbol{A}(t))$ and $I(\boldsymbol{Q}(t))=I(\boldsymbol{A}(t))^{\perp}$.

Substituting the space decompositions Eqs. (5) – (6) into Eq. (2), a new DAE can be obtained.

Lemma 1 The descriptor system (2) can be reformulated as a new DAE

$$\left.\begin{aligned}\boldsymbol{x}_1'(t)&=\boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11}-\boldsymbol{A}_1')\boldsymbol{x}_1(t)+\boldsymbol{A}_1^{-1}\boldsymbol{C}_{12}\boldsymbol{x}_2(t)+\\&\quad\ \boldsymbol{A}_1^{-1}\boldsymbol{B}_1\boldsymbol{u}(t)\\ \boldsymbol{0}&=\boldsymbol{C}_{21}\boldsymbol{x}_1(t)+\boldsymbol{C}_{22}\boldsymbol{x}_2(t)+\boldsymbol{B}_2\boldsymbol{u}(t)\end{aligned}\right\} \tag{7}$$

with respect to the decompositions (5) – (6), where $\boldsymbol{x}_1(t)=(\boldsymbol{I}-\boldsymbol{P}(t))\boldsymbol{x}(t), \boldsymbol{x}_2(t)=\boldsymbol{P}(t)\boldsymbol{x}(t)$, and

$$\left.\begin{aligned}\boldsymbol{A}_1&=P_{I(\boldsymbol{A}(t))}\boldsymbol{A}(t)|_{K(\boldsymbol{A}(t))^{\perp}}\\ \boldsymbol{B}_1&=P_{I(\boldsymbol{A}(t))}\boldsymbol{B}(t)\\ \boldsymbol{B}_2&=P_{I(\boldsymbol{A}(t))^{\perp}}\boldsymbol{B}(t)\\ \boldsymbol{C}_{11}&=P_{I(\boldsymbol{A}(t))}\boldsymbol{C}(t)|_{K(\boldsymbol{A}(t))^{\perp}}\\ \boldsymbol{C}_{12}&=P_{I(\boldsymbol{A}(t))}\boldsymbol{C}(t)|_{K(\boldsymbol{A}(t))}\\ \boldsymbol{C}_{21}&=P_{I(\boldsymbol{A}(t))^{\perp}}\boldsymbol{C}(t)|_{K(\boldsymbol{A}(t))^{\perp}}\\ \boldsymbol{C}_{22}&=P_{I(\boldsymbol{A}(t))^{\perp}}\boldsymbol{C}(t)|_{K(\boldsymbol{A}(t))}\end{aligned}\right\} \tag{8}$$

Proof Through the space decomposition, the system (2) can be expressed as $\left(\begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{pmatrix}\right)' = \begin{pmatrix} \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_1(t) \\ \boldsymbol{x}_2(t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{pmatrix}\boldsymbol{u}(t)$, i. e., $(\boldsymbol{A}_1\boldsymbol{x}_1(t))' = \boldsymbol{C}_{11}\boldsymbol{x}_1(t) + \boldsymbol{C}_{12}\boldsymbol{x}_2(t) + \boldsymbol{B}_1\boldsymbol{u}(t)$ and $\boldsymbol{0} = \boldsymbol{C}_{21}\boldsymbol{x}_1(t) + \boldsymbol{C}_{22}\boldsymbol{x}_2(t) + \boldsymbol{B}_2\boldsymbol{u}(t)$, where the involved operator-valued functions are defined in Eq. (8). From the invertibility of $\boldsymbol{A}_1$, $\boldsymbol{x}_1'(t) = \boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \boldsymbol{A}_1')\boldsymbol{x}_1(t) + \boldsymbol{A}_1^{-1}\boldsymbol{C}_{12}\boldsymbol{x}_2(t) + \boldsymbol{A}_1^{-1}\boldsymbol{B}_1\boldsymbol{u}(t)$.

Remark 1 By space decompositions, the system (2) with higher dimension can be reduced to a lower dimensional system, which is easy to check and compute.

For $\boldsymbol{V}$, $\boldsymbol{W}(t)$, $\boldsymbol{S}(t)$, $\boldsymbol{R}(t)$ and $\boldsymbol{P}(t)$, $\boldsymbol{Q}(t)$, a natural assumption is given as follows.

Assumption 1

1) $\boldsymbol{V}$ is positive semi-definite;

2) The operator matrix $\begin{pmatrix} \boldsymbol{W}(t) & \boldsymbol{S}(t) \\ \boldsymbol{S}^*(t) & \boldsymbol{R}(t) \end{pmatrix} \in L(X \oplus Z)$ is positive semi-definite for all $t \in [0, T]$;

3) $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ are continuous.

Based on Lemma 1 and Assumption 1, a sufficient condition solving the OCP for system (2) – (4) can be found.

Theorem 1 Under the conditions 1) and 2) in Assumption 1, if the pair of continuous functions $(\tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t)): [0, T] \to X \times Z$ satisfies the following equations

$$\left.\begin{aligned} &(\boldsymbol{A}(t)\boldsymbol{x}(t))' = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) = \boldsymbol{y}_0 \\ &\boldsymbol{A}(0)\boldsymbol{x}(0) \end{aligned}\right\} \tag{9}$$

$$\boldsymbol{W}(t)\boldsymbol{x}(t) + \boldsymbol{S}(t)\boldsymbol{u}(t) = -(\boldsymbol{V}\boldsymbol{x}(t))' + \begin{pmatrix} (\boldsymbol{A}_1(\boldsymbol{C}_{11} - \boldsymbol{A}_1'))^* & 0 \\ (\boldsymbol{A}_1^{-1}\boldsymbol{C}_{12})^* & -\frac{\mathrm{d}}{\mathrm{d}t} \end{pmatrix}(-\boldsymbol{V}\boldsymbol{x}(t)) \tag{10}$$

$$\boldsymbol{S}^*(t)\boldsymbol{x}(t) + \boldsymbol{R}(t)\boldsymbol{u}(t) = (\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*(-\boldsymbol{V}_{11} \ \ -\boldsymbol{V}_{12})\boldsymbol{x}(t) \tag{11}$$

$$\boldsymbol{V}_{12}^*\boldsymbol{x}_1(0) + \boldsymbol{V}_{22}\boldsymbol{x}_2(0) = \boldsymbol{V}_{12}^*\boldsymbol{x}_1(T) + \boldsymbol{V}_{22}\boldsymbol{x}_2(T) = \boldsymbol{0} \tag{12}$$

then $\tilde{\boldsymbol{u}}(t)$ can be regarded as an optimal control for the problem (2) – (4), where $\boldsymbol{A}_1$, $\boldsymbol{B}$, $\boldsymbol{C}_{11}$, $\boldsymbol{C}_{12}$ are defined in Eq. (8), and $\boldsymbol{V}_{11}$, $\boldsymbol{V}_{12}$, $\boldsymbol{V}_{12}^*$, $\boldsymbol{V}_{22}$ are the entries of $\boldsymbol{V}$ as an operator on $K(\boldsymbol{A}(t))^{\perp} \oplus K(\boldsymbol{A}(t))$.

Proof Let $\tilde{\boldsymbol{x}}(t)$, $\tilde{\boldsymbol{u}}(t)$ satisfy the system (9) – (12), and $\boldsymbol{u}(t)$, $\boldsymbol{x}(t)$ be an arbitrary admissible control input and the corresponding solution of the problem (2) – (3), respectively. $J(\boldsymbol{u}(t), \boldsymbol{x}(t)) - J(\tilde{\boldsymbol{u}}(t), \tilde{\boldsymbol{x}}(t)) = \frac{1}{2}\langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}(\boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T))\rangle + \boldsymbol{U} + \frac{1}{2}\int_0^T \left\langle \begin{pmatrix} \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t) \\ \boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t) \end{pmatrix}, \begin{pmatrix} \boldsymbol{W}(t) & \boldsymbol{S}(t) \\ \boldsymbol{S}^*(t) & \boldsymbol{R}(t) \end{pmatrix}\begin{pmatrix} \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t) \\ \boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t) \end{pmatrix} \right\rangle \mathrm{d}t$ can be got from Eq. (4), here $\boldsymbol{U} = \int_0^T (\langle \boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t), \boldsymbol{S}^*(t)\tilde{\boldsymbol{x}}(t) + \boldsymbol{R}(t)\tilde{\boldsymbol{u}}(t)\rangle + \langle \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t), \boldsymbol{W}(t)\tilde{\boldsymbol{x}}(t) + \boldsymbol{S}(t)\tilde{\boldsymbol{u}}(t)\rangle)\mathrm{d}t + \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle$.

From the conditions 1) and 2) in Assumption 1, if $\boldsymbol{U} = 0$, the inequality $J(\boldsymbol{u}(t), \boldsymbol{x}(t)) - J(\tilde{\boldsymbol{u}}(t), \tilde{\boldsymbol{x}}(t)) \geqslant 0$ can be obtained, and then $\tilde{\boldsymbol{u}}(t)$ is the optimal control. In the following, $\boldsymbol{U} = \boldsymbol{0}$ will be shown by using Eqs. (9) – (12). In fact,

$$\begin{aligned} \boldsymbol{U} = {} & \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle + \int_0^T \Big(\langle \boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t), \\ & (\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*(-\boldsymbol{V}_{11} \ \ -\boldsymbol{V}_{12})\tilde{\boldsymbol{x}}(t)\rangle + \Big\langle \boldsymbol{x}(t) - \\ & \tilde{\boldsymbol{x}}(t), -(\boldsymbol{V}\tilde{\boldsymbol{x}}(t))' - \begin{pmatrix} (\boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \boldsymbol{A}_1'))^* & \boldsymbol{0} \\ (\boldsymbol{A}_1^{-1}\boldsymbol{C}_{12})^* & -\frac{\mathrm{d}}{\mathrm{d}t} \end{pmatrix}\boldsymbol{V}\tilde{\boldsymbol{x}}(t) \Big\rangle \Big)\mathrm{d}t = \\ & \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle + \int_0^T \Big(\langle \boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \\ & \boldsymbol{A}_1')(\boldsymbol{x}_1(t) - \tilde{\boldsymbol{x}}_1(t)) + \boldsymbol{A}_1^{-1}\boldsymbol{C}_{12}(\boldsymbol{x}_2(t) - \tilde{\boldsymbol{x}}_2(t)) + \\ & \boldsymbol{A}_1^{-1}\boldsymbol{B}_1(\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)), -\boldsymbol{V}_{11}\tilde{\boldsymbol{x}}_1(t) - \boldsymbol{V}_{12}\tilde{\boldsymbol{x}}_2(t)\rangle + \\ & \langle \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t), -(\boldsymbol{V}\tilde{\boldsymbol{x}}(t))'\rangle - \Big\langle \boldsymbol{x}_2(t) - \\ & \tilde{\boldsymbol{x}}_2(t), \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)(\boldsymbol{V}_{12}^*\tilde{\boldsymbol{x}}_1(t) + \boldsymbol{V}_{22}\tilde{\boldsymbol{x}}_2(t)) \Big\rangle \Big)\mathrm{d}t = \\ & \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle + \int_0^T (\langle \boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \\ & \boldsymbol{A}_1')(\boldsymbol{x}_1(t) - \tilde{\boldsymbol{x}}_1(t)) + \boldsymbol{A}_1^{-1}\boldsymbol{C}_{12}(\boldsymbol{x}_2(t) - \tilde{\boldsymbol{x}}_2(t)) + \\ & \boldsymbol{A}_1^{-1}\boldsymbol{B}_1(\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)), -\boldsymbol{V}_{11}\tilde{\boldsymbol{x}}_1(t) - \boldsymbol{V}_{12}\tilde{\boldsymbol{x}}_2(t)\rangle + \\ & \langle \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t), -(\boldsymbol{V}\tilde{\boldsymbol{x}}(t))'\rangle + \langle (\boldsymbol{x}_2(t) - \tilde{\boldsymbol{x}}_2(t))', \\ & -\boldsymbol{V}_{12}\tilde{\boldsymbol{x}}_1(t) - \boldsymbol{V}_{22}\tilde{\boldsymbol{x}}_2(t))\rangle)\mathrm{d}t + \langle \boldsymbol{x}_2(T) - \end{aligned}$$

$$\tilde{\boldsymbol{x}}_2(T), \boldsymbol{V}_{12}^*\tilde{\boldsymbol{x}}_1(T) + \boldsymbol{V}_{22}\tilde{\boldsymbol{x}}_2(T)\rangle - \langle \boldsymbol{x}_2(0) - \tilde{\boldsymbol{x}}_2(0), \boldsymbol{V}_{12}^*\tilde{\boldsymbol{x}}_1(0) + \boldsymbol{V}_{22}\tilde{\boldsymbol{x}}_2(0)\rangle,$$

then

$$\begin{aligned}
\boldsymbol{U} = {} & \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle + \int_0^T (\langle (\boldsymbol{x}_1(t) - \\
& \tilde{\boldsymbol{x}}_1(t))', -\boldsymbol{V}_{11}\tilde{\boldsymbol{x}}_1(t) - \boldsymbol{V}_{12}\tilde{\boldsymbol{x}}_2(t)\rangle + \\
& \langle \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t), (-\boldsymbol{V}\tilde{\boldsymbol{x}}(t))'\rangle + \langle (\boldsymbol{x}_2(t) - \\
& \tilde{\boldsymbol{x}}_2(t))', -\boldsymbol{V}_{12}^*\tilde{\boldsymbol{x}}_1(t) - \boldsymbol{V}_{22}\tilde{\boldsymbol{x}}_2(t)\rangle)\mathrm{d}t = \\
& \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle + \int_0^T (\langle (\boldsymbol{x}(t) - \\
& \tilde{\boldsymbol{x}}(t))', -\boldsymbol{V}\tilde{\boldsymbol{x}}(t)\rangle + \langle \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t), \\
& (-\boldsymbol{V}\tilde{\boldsymbol{x}}(t))'\rangle)\mathrm{d}t = \\
& \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle + \\
& \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t), -\boldsymbol{V}\tilde{\boldsymbol{x}}(t)\rangle \mathrm{d}t = \\
& \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle + \langle \boldsymbol{x}(T) - \tilde{\boldsymbol{x}}(T), \\
& -\boldsymbol{V}\tilde{\boldsymbol{x}}(T)\rangle + \langle \boldsymbol{x}(0) - \tilde{\boldsymbol{x}}(0), \boldsymbol{V}\tilde{\boldsymbol{x}}(0)\rangle = \\
& \langle \boldsymbol{x}(0) - \tilde{\boldsymbol{x}}(0), \boldsymbol{V}\tilde{\boldsymbol{x}}(0)\rangle
\end{aligned}$$

can be obtained from Lemma 1 and Eq. (12). Then $\boldsymbol{x}_1(0) = \tilde{\boldsymbol{x}}_1(0)$ is obtained from Eq. (9) and the invertibility of $\boldsymbol{A}_1$, therefore, $\boldsymbol{U} = \langle \boldsymbol{x}_2(0) - \tilde{\boldsymbol{x}}_2(0), \boldsymbol{V}_{12}^*\tilde{\boldsymbol{x}}_1(0) + \boldsymbol{V}_{22}\tilde{\boldsymbol{x}}_2(0)\rangle$, and then $\boldsymbol{U} = 0$ follows from Eq. (12) immediately.

Theorem 1 provides a skillful method solving the OCP for system (2) - (4) by space decompositions and Lemma 1. In contrast to the result in Ref. [17], Theorem 1 does not need to introduce extra variables, and it will make readers easier to design the OCP. The solvability of the problem (2) - (4) under the restriction of conditions (9) - (12) needs to discuss further yet.

4 Solvability

In Ref. [17], it is necessary to satisfy $I(\boldsymbol{V}) \subseteq I(\boldsymbol{A}^*(T))$ and the invertibility of the 3×3 operator matrix for solvability of the OCP, however, the two conditions are difficult to meet for general time-varying dynamic systems. For this, the solvability of Eqs. (2) - (4) will be considered without these two limitations.

Theorem 2 Under Assumption 1, for $t \in [0, T]$, if $\boldsymbol{V}_{12}$ and $\boldsymbol{R}(t)$ are invertible on $K(\boldsymbol{A}(t))^{\perp}$ and Z, respectively, and $\boldsymbol{E}(t)$ is positive semi-definite, then the problem (2) - (4) is solvable, where $\boldsymbol{E}(t) = \begin{pmatrix} \boldsymbol{V}_{11} \\ \boldsymbol{V}_{12}^* \end{pmatrix}(\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)\boldsymbol{R}^{-1}(\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*(\boldsymbol{V}_{11} \quad \boldsymbol{V}_{12}) + \boldsymbol{W}(t) - \boldsymbol{S}(t)\boldsymbol{R}^{-1}(t)\boldsymbol{S}^*(t)$, and $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, $\boldsymbol{V}_{11}$, $\boldsymbol{V}_{12}$ are defined as in Lemma 1 and Theorem 1, respectively.

Proof From Lemma 1, the problem (2) - (3) can be rewritten as Eq. (7) and $\boldsymbol{A}_1(0)\boldsymbol{x}_1(0) = \boldsymbol{y}_{01}$ with $\boldsymbol{y}_{01} = P_{I(\boldsymbol{A}(t))}\boldsymbol{y}_0$. Make the block representations

$$\left.\begin{aligned}
& \boldsymbol{W}(t) = \begin{pmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \\ \boldsymbol{W}_{12}^* & \boldsymbol{W}_{22} \end{pmatrix}: K(\boldsymbol{A}(t))^{\perp} \oplus K(\boldsymbol{A}(t)) \rightarrow \\
& \qquad K(\boldsymbol{A}(t))^{\perp} \oplus K(\boldsymbol{A}(t)) \\
& \boldsymbol{S}(t) = \begin{pmatrix} \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{pmatrix}: Z \rightarrow K(\boldsymbol{A}(t))^{\perp} \oplus K(\boldsymbol{A}(t))
\end{aligned}\right\} \tag{13}$$

so that Eq. (10) can be decomposed into

$$\begin{aligned}
-\boldsymbol{V}_{11}\boldsymbol{x}_1'(t) - \boldsymbol{V}_{12}\boldsymbol{x}_2'(t) = {} & (\boldsymbol{W}_{11} + (\boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \boldsymbol{A}_1'))^* \cdot \\
& \boldsymbol{V}_{11})\boldsymbol{x}_1(t) + (\boldsymbol{W}_{12} + (\boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \boldsymbol{A}_1'))^*\boldsymbol{V}_{12}) \cdot \\
& \boldsymbol{x}_2(t) + \boldsymbol{S}_1\boldsymbol{u}(t)
\end{aligned} \tag{14}$$

$$\begin{aligned}
\boldsymbol{0} = {} & (\boldsymbol{W}_{12}^* + (\boldsymbol{A}_1^{-1}\boldsymbol{C}_{12})^*\boldsymbol{V}_{11})\boldsymbol{x}_1(t) + \boldsymbol{S}_2\boldsymbol{u}(t) + \\
& (\boldsymbol{W}_{22} + (\boldsymbol{A}_1^{-1}\boldsymbol{C}_{12})^*\boldsymbol{V}_{12})\boldsymbol{x}_2(t)
\end{aligned} \tag{15}$$

Since $\boldsymbol{R}(t)$ is invertible, $\boldsymbol{u}(t) = -\boldsymbol{R}^{-1}(t)(\boldsymbol{S}_1^*\boldsymbol{x}_1(t) + \boldsymbol{S}_2^*\boldsymbol{x}_2(t)) - \boldsymbol{R}^{-1}(t)(\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*(\boldsymbol{V}_{11}\boldsymbol{x}_1(t) + \boldsymbol{V}_{12}\boldsymbol{x}_2(t))$ can be obtained from Eq. (11). Then Eqs. (7), (14) and (15) can be rewritten as

$$\left.\begin{aligned}
\boldsymbol{x}_1'(t) = {} & \boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \boldsymbol{A}_1')\boldsymbol{x}_1(t) + \boldsymbol{A}_1^{-1}\boldsymbol{C}_{12}\boldsymbol{x}_2(t) - \\
& \boldsymbol{A}_1^{-1}\boldsymbol{B}_1\boldsymbol{R}^{-1}(t)(\boldsymbol{S}_1^*\boldsymbol{x}_1(t) + \boldsymbol{S}_2^*\boldsymbol{x}_2(t)) - \\
& \boldsymbol{A}_1^{-1}\boldsymbol{B}_1\boldsymbol{R}^{-1}(t)(\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*(\boldsymbol{V}_{11}\boldsymbol{x}_1(t) + \boldsymbol{V}_{12}\boldsymbol{x}_2(t)) \\
\boldsymbol{0} = {} & -\boldsymbol{B}_2\boldsymbol{R}^{-1}(t)(\boldsymbol{S}_1^*\boldsymbol{x}_1(t) + \boldsymbol{S}_2^*\boldsymbol{x}_2(t)) - \\
& \boldsymbol{B}_2\boldsymbol{R}^{-1}(t)(\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*(\boldsymbol{V}_{11}\boldsymbol{x}_1(t) + \boldsymbol{V}_{12}\boldsymbol{x}_2(t)) + \\
& \boldsymbol{C}_{12}\boldsymbol{x}_1(t) + \boldsymbol{C}_{22}\boldsymbol{x}_2(t)
\end{aligned}\right\} \tag{16}$$

$$\begin{aligned}
-\boldsymbol{V}_{11}\boldsymbol{x}_1'(t) - \boldsymbol{V}_{12}\boldsymbol{x}_2'(t) = {} & \\
& -\boldsymbol{S}_1\boldsymbol{R}^{-1}(t)(\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*\boldsymbol{V}_{11}\boldsymbol{x}_1(t) - \\
& \boldsymbol{S}_1\boldsymbol{R}^{-1}(t)(\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*\boldsymbol{V}_{12}\boldsymbol{x}_2(t)) - \\
& \boldsymbol{S}_1\boldsymbol{R}^{-1}(t)(\boldsymbol{S}_1^*\boldsymbol{x}_1(t) + \boldsymbol{S}_2^*\boldsymbol{x}_2(t)) + \\
& (\boldsymbol{W}_{11} + (\boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \boldsymbol{A}_1'))^*\boldsymbol{V}_{11})\boldsymbol{x}_1(t) + \\
& (\boldsymbol{W}_{12} + (\boldsymbol{A}_1^{-1}(\boldsymbol{C}_{11} - \boldsymbol{A}_1'))^*\boldsymbol{V}_{12})\boldsymbol{x}_2(t)
\end{aligned} \tag{17}$$

$$\begin{aligned}
\boldsymbol{0} = {} & -\boldsymbol{S}_2\boldsymbol{R}^{-1}(t)(\boldsymbol{A}_1^{-1}\boldsymbol{B}_1)^*(\boldsymbol{V}_{11}\boldsymbol{x}_1(t) + \boldsymbol{V}_{12}\boldsymbol{x}_2(t)) - \\
& \boldsymbol{S}_2\boldsymbol{R}^{-1}(t)(\boldsymbol{S}_1^*\boldsymbol{x}_1(t) + \boldsymbol{S}_2^*\boldsymbol{x}_2(t)) +
\end{aligned}$$

$$(W_{12}^* + (A_1^{-1}C_{12})^* V_{11})x_1(t) + (W_{22} + (A_1^{-1}C_{12})^* V_{12})x_2(t) \quad (18)$$

respectively. Scalarly multiply the first equation of Eq. (16) by $V_{11}x_1(t) + V_{12}x_2(t)$, Eq. (17) by $-x_1(t)$ and Eq. (18) by $-x_2(t)$, and add the results, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x_1(t), V_{11}x_1(t) + V_{12}x_2(t)\rangle = \langle R^{-1}(t)(S_1^* x_1(t) + S_2^* x_2(t)), S_1^* x_1(t) + S_2^* x_2(t)\rangle - \langle W(t)x(t), x(t)\rangle - \langle R^{-1}(t)(A_1^{-1}B_1)^*(V_{11}x_1(t) + V_{12}x_2(t)), (A_1^{-1}B_1)^*(V_{11}x_1(t) + V_{12}x_2(t))\rangle \quad (19)$$

Let $x_1(0) = V_{11}x_1(T) + V_{12}x_2(T) = \mathbf{0}$. Integrating Eq. (19) on $[0,T]$, in combination with the boundary conditions means that $\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\langle x_1(t), V_{11}x_1(t) + V_{12}x_2(t)\rangle \mathrm{d}t = 0$, i. e.,

$$\int_0^T \langle R^{-1}(t)(S_1^* x_1(t) + S_2^* x_2(t)), S_1^* x_1(t) + S_2^* x_2(t)\rangle - \langle W(t)x(t), x(t)\rangle - \langle R^{-1}(t)(A_1^{-1}B_1)^*(V_{11}x_1(t) + V_{12}x_2(t)), (A_1^{-1}B_1)^*(V_{11}x_1(t) + V_{12}x_2(t))\rangle \mathrm{d}t = -\int_0^T \langle (Ex)(t), x(t)\rangle \mathrm{d}t = 0 \quad (20)$$

Because $E(t)$ is positive semi-definite and continuous with respect to t, it follows from Eq. (20) that $E(t)x(t) \equiv \mathbf{0}$, i. e.,

$$(W_{11} - S_1R^{-1}(t)S_1^*)x_1(t) + (W_{12} - S_1R^{-1}(t)S_2^*)x_2(t) = -V_{11}(A_1^{-1}B_1)R^{-1}(t)(A_1^{-1}B_1)^*(V_{11}x_1(t) + V_{12}x_2(t)) \quad (21)$$

Then $V_{11}x_1(t) + V_{12}x_2(t) \equiv \mathbf{0}$ according to Eqs. (17), (21) and the boundary condition $V_{11}x_1(T) + V_{12}x_2(T) = \mathbf{0}$. So $x(t) \equiv 0$ can be obtained by the first equation of Eq. (16) and the invertibility of V_{12} on $K(A(t))^\perp$, that is, the system (8) – (12) with boundary condition $x_1(0) = V_{11}x_1(T) + V_{12}x_2(T) = \mathbf{0}$ is uniquely solvable. Therefore, the problem (2) – (4) is solvable.

In view of $V_{11}x_1(t) + V_{12}x_2(t) \equiv \mathbf{0}$, from the second equation of Eq. (16), the following similar result still holds when $B_2R^{-1}(t)S_2^* - C_{22}$ is invertible.

Theorem 3 Under Assumption 1, if $R(t)$ and $B_2R^{-1}(t)S_2^* - C_{22}$ are invertible for $t \in [0,T]$, and $E(t)$ is positive semi-definite, then the problem (2) – (4) is solvable, where $B_2, C_{22}, E(t)$ and S_2 are defined as in Lemma 1, Theorem 1 and the expression (13), respectively.

Note from Theorems 2 and 3 that some sufficient conditions for the solvability of the problem (2) – (4) under the restriction of conditions (9) – (12), without $I(V) \subseteq I(A^*(T))$ and the invertibility of the 3×3 operator matrix are obtained. However, Theorems 2 and 3 still require some constraints on the operator V, and thus we are devoted to dropping these constraints.

Theorem 4 Under Assumption 1, if $R(t)$ and $W_{12} - S_1R^{-1}(t)S_2^*$ (or V_{12}) are invertible for $t \in [0, T]$, and $F(t) = W(t) - S(t)R^{-1}(t)S^*(t)$ is positive semi-definite, then the problem (2) – (4) is solvable, where W_{12}, S_1, S_2 are defined as in Eq. (13).

Proof According to the expression of $F(t)$, Eq. (20) can be rewritten as

$$-\int_0^T \langle R^{-1}(t)(A_1^{-1}B_1)^*(V_{11}x_1(t) + V_{12}x_2(t)), (A_1^{-1}B_1)^*(V_{11}x_1(t) + V_{12}x_2(t))\rangle + \langle F(t)x(t), x(t)\rangle \mathrm{d}t = \mathbf{0} \quad (22)$$

with $x_1(0) = V_{11}x_1(T) + V_{12}x_2(T) = \mathbf{0}$. Since $F(t)$, $R^{-1}(t)$ are positive semi-definite and continuous with respect to t, it follows from Eq. (22) that $F(t)x(t) \equiv \mathbf{0}$ and $R^{-1}(t)(A_1^{-1}B_1)^*(V_{11}x_1(t) + V_{12}x_2(t)) \equiv \mathbf{0}$, then

$$(W_{11} - S_1R^{-1}(t)S_1^*)x_1(t) + (W_{12} - S_1R^{-1}(t)S_2^*)x_2(t) \equiv \mathbf{0} \quad (23)$$

$$(W_{12}^* - S_2R^{-1}(t)S_1^*)x_1(t) + (W_{22} - S_2R^{-1}(t)S_2^*)x_2(t) \equiv \mathbf{0} \quad (24)$$

From Eqs. (16), (23) and the invertibility of $W_{12} - S_1R^{-1}(t)S_2^*$ (or from Eqs. (16), (17), (23) and the invertibility of V_{12}), it follows that $x(t) \equiv \mathbf{0}$, whence the problem (2) – (4) is solvable.

Similar to Theorem 4, the following result can be obtained.

Theorem 5 Under Assumption 1, if $R(t)$ and $W_{22} - S_2R^{-1}(t)S_2^*$ are invertible for $t \in [0,T]$, and $F(t)$ is positive semi-definite, then the problem (2) – (4) is solvable, where W_{22}, S_2 are defined as in Eq. (13).

5 Examples

This section provides some interesting examples to demonstrate the validity of theoretical results and compares with those of Ref. [17].

Example 1 Let $X = Y = L^2[0,T] \oplus L^2[0,T]$, $Z = L^2[0,T]$, $\boldsymbol{A}(t) = \begin{pmatrix} \boldsymbol{0} & t\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}$, $\boldsymbol{C}(t) \equiv \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$, $\boldsymbol{B}(t) \equiv \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{pmatrix}$, $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$, $\boldsymbol{W}(t) \equiv \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$, $\boldsymbol{S}(t) \equiv (\boldsymbol{0}, \boldsymbol{0})^{\mathrm{T}}$, $\boldsymbol{R}(t) \equiv \boldsymbol{I}$, $\boldsymbol{y}_0 = (0, \boldsymbol{\eta}_2^{\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{\eta}_2 \neq \boldsymbol{0}$. It is obvious that $K(\boldsymbol{A}(t)) = \{(\boldsymbol{f}_1^{\mathrm{T}}, \boldsymbol{0})^{\mathrm{T}} : \boldsymbol{f}_1 \in L^2[0,T]\}$, $I(\boldsymbol{A}(t)) = \{(t\boldsymbol{f}_2^{\mathrm{T}}, \boldsymbol{f}_2^{\mathrm{T}})^{\mathrm{T}} : \boldsymbol{f}_2 \in L^2[0,T]\}$, $\boldsymbol{P}(t) \equiv \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$ and $\boldsymbol{Q}(t) = \dfrac{1}{1+t^2} \begin{pmatrix} \boldsymbol{I} & -t\boldsymbol{I} \\ -t\boldsymbol{I} & t^2\boldsymbol{I} \end{pmatrix}$. By computing, $\boldsymbol{W}_{22}$ is invertible on $K(\boldsymbol{A}(t))$ and $\boldsymbol{F}(t) = \boldsymbol{W}(t)$ is positive semi-definite, which satisfies the conditions of Theorem 5 and then the optimal control for Eqs. (2) – (4) exists. In the following, the optimal control $\tilde{\boldsymbol{u}}(t)$ can be found. In fact, the system (9) – (12) becomes

$$\left(\begin{pmatrix} \boldsymbol{0} & t\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \boldsymbol{x}(t) \right)' = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{pmatrix} \boldsymbol{u}(t) \tag{25}$$

$$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \boldsymbol{x}(0) = \boldsymbol{y}_0 = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\eta}_2 \end{pmatrix} \tag{26}$$

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \boldsymbol{u}(t) = 0 \tag{27}$$

$$(\boldsymbol{0} \quad \boldsymbol{0}) \boldsymbol{x}(t) + \boldsymbol{u}(t) = 0 \tag{28}$$

Eqs. (27) and (28) immediately lead to $\boldsymbol{\xi}_1(t) = \boldsymbol{u}(t) \equiv \boldsymbol{0}$, where $\boldsymbol{x}(t) = (\boldsymbol{\xi}_1(t)^{\mathrm{T}}, \boldsymbol{\xi}_2(t)^{\mathrm{T}})^{\mathrm{T}} \in L^2[0,T] \oplus L^2[0,T]$. It follows from Eq. (25) that $\boldsymbol{\xi}_2'(t) = \boldsymbol{0}$, which together with the initial condition (26) gives $\boldsymbol{\xi}_2(t) \equiv \boldsymbol{\eta}_2$. Consequently, the problem (9) – (12) has the unique solution $\tilde{\boldsymbol{x}}(t) \equiv \boldsymbol{y}_0$, $\tilde{\boldsymbol{u}}(t) \equiv \boldsymbol{0}$, and then $\tilde{\boldsymbol{u}}(t) \equiv 0$ is the optimal control for Eqs. (2) – (4).

In addition, some image simulations are used to verify the authenticity of the obtained results. For Eqs. (2) – (3), $\boldsymbol{\xi}_2(t) \equiv \boldsymbol{\eta}_2$ and $\tilde{\boldsymbol{u}}(t) \equiv \boldsymbol{0}$ can be gained, then the associated quadratic cost functional can be expressed as $J = \dfrac{1}{2} \displaystyle\int_0^T \boldsymbol{\xi}_1^2(t)\,\mathrm{d}t$, where $\boldsymbol{\xi}_1(t)$ is any function about t. Because of the arbitrariness of $\boldsymbol{\xi}_1(t)$, $T = 10$ and three special functions $\sin t$, $\cos t$, $t^2 + 1$ can be chosen without loss of generality.

Obviously, the area of blue part in Fig. 1 is $\displaystyle\int_0^T \boldsymbol{\xi}_1^2(t)\,\mathrm{d}t$ and J equals 2.385 9, 2.614 1, 10 338, respectively. Further, $J \geqslant 0$ can be obtained for any function $\boldsymbol{\xi}_1(t)$. Thus, from the minimum $J = 0$, $\boldsymbol{\xi}_1(t) \equiv \boldsymbol{0}$ can be got. Therefore, the optimal control and relevant solution $\tilde{\boldsymbol{u}}(t) \equiv 0$, $\tilde{\boldsymbol{x}}(t) \equiv \boldsymbol{y}_0$ can be obtained.

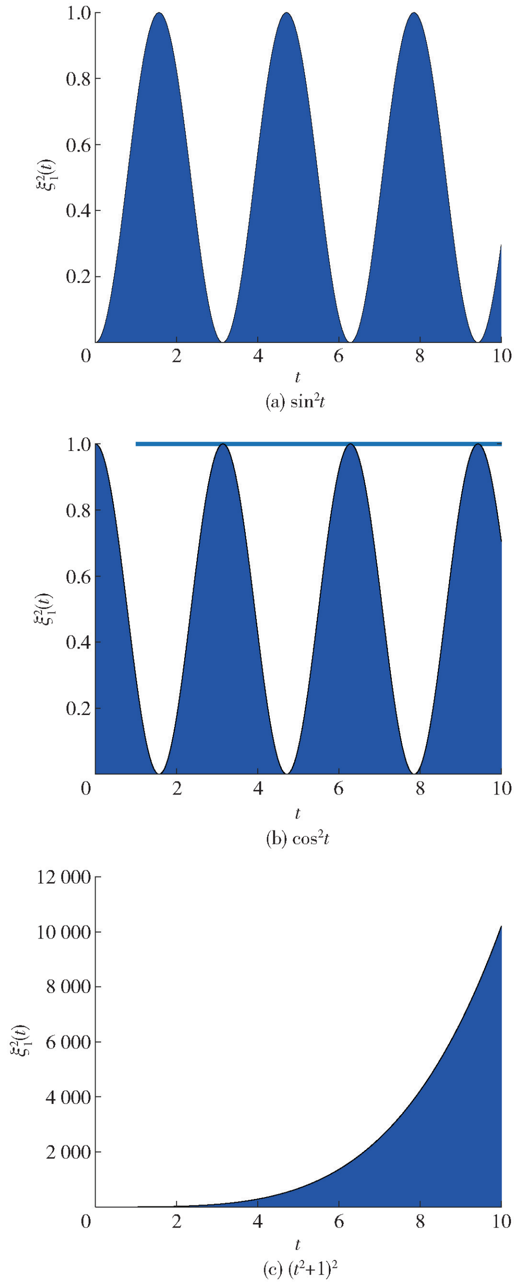


Fig. 1 Images of $\boldsymbol{\xi}_1^2(t)$ by three special functions in Example 1

In Ref. [17], the condition $I(\boldsymbol{V}) \subseteq I(\boldsymbol{A}^*(T))$ and the invertibility of 3×3 operator matrix are necessary, but those are unnecessary and easy to check for Example 1 in this work. In fact, we can solve the problems if these conditions fail as follows.

Example 2 Choose $V=\alpha I$, $\alpha>0$, $\eta_2=0$ in Example 1,

$$A_1=\begin{pmatrix}0 & tI\\ 0 & I\end{pmatrix}\Big|_{K(A(t))^{\perp}},A_1^{-1}=\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}\Big|_{I(A(t))},$$

$$C_{11}=\begin{pmatrix}0 & \frac{t^2}{1+t^2}I\\ 0 & \frac{t}{1+t^2}I\end{pmatrix}\Bigg|_{K(A(t))^{\perp}},$$

$$C_{12}=\begin{pmatrix}0 & \frac{t^2}{1+t^2}I\\ 0 & \frac{t}{1+t^2}I\end{pmatrix}\Bigg|_{K(A(t))},$$

$$C_{22}=\begin{pmatrix}0 & \frac{1}{1+t^2}I\\ 0 & -\frac{t}{1+t^2}I\end{pmatrix}\Bigg|_{K(A(t))},$$

$$B_1=\begin{pmatrix}\frac{t^2}{1+t^2}I\\ \frac{t}{1+t^2}I\end{pmatrix}, B_2=\begin{pmatrix}\frac{1}{1+t^2}I\\ \frac{-t}{1+t^2}I\end{pmatrix},$$

$$V_{11}=\begin{pmatrix}0 & 0\\ 0 & \alpha I\end{pmatrix}\Big|_{K(A(t))^{\perp}}, V_{12}=\begin{pmatrix}0 & 0\\ 0 & \alpha I\end{pmatrix}\Big|_{K(A(t))},$$

$$V_{22}=\begin{pmatrix}\alpha I & 0\\ 0 & 0\end{pmatrix}\Big|_{K(A(t))}, W_{11}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\Big|_{K(A(t))^{\perp}},$$

$$W_{12}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\Big|_{K(A(t))}, W_{22}=\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}\Big|_{K(A(t))}.$$

Obviously, $A(t)$ may not be invertible, but A_1 as an operator from $K(A(t))^{\perp}$ to $I(A(t))$ is invertible. So we can deal with this example without the range inclusion condition and the invertibility of $G(t)$. Similarly, the system (9) – (12) can be used to solve the problem of minimizing the function (4) with the system (2) – (3). We now find the optimal control $\tilde{u}(t)$. In fact, Eq. (9) is equivalent to $\left(\begin{pmatrix}0 & tI\\ 0 & I\end{pmatrix}x(t)\right)'=\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}x(t)+\begin{pmatrix}I\\ 0\end{pmatrix}u(t)$ and $\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}x(0)=y_0=\begin{pmatrix}0\\ 0\end{pmatrix}$. It means $\xi_1(0)=\xi_2(t)=u(t)\equiv 0$, where $x(t)=(\xi_1(t)^{\mathrm{T}},\xi_2(t)^{\mathrm{T}})^{\mathrm{T}}\in L^2[0,T]\oplus L^2[0,T]$. Since $x_1(t)=(0,\xi_2(t)^{\mathrm{T}})^{\mathrm{T}}, x_2(t)=(\xi_1(t)^{\mathrm{T}},0)^{\mathrm{T}}$, it follows that $(A_1^{-1}(C_{11}-A_1'))^{*}=\frac{1}{1+t^2}\left(\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}0 & -I\\ 0 & tI\end{pmatrix}\right)^{*}=\frac{1}{1+t^2}\begin{pmatrix}0 & 0\\ 0 & tI\end{pmatrix}, (A_1^{-1}C_{12})^{*}=\frac{1}{1+t^2}\left(\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}0 & -t^2I\\ 0 & tI\end{pmatrix}\right)^{*}=\frac{1}{1+t^2}\begin{pmatrix}0 & 0\\ 0 & tI\end{pmatrix}, Vx(t)=\begin{pmatrix}0&0&0&0\\0&\alpha I&0&\alpha I\\0&0&\alpha I&0\\0&\alpha I&0&0\end{pmatrix}\begin{pmatrix}0\\ \xi_2(t)\\ \xi_1(t)\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \alpha\xi_2(t)\\ \alpha\xi_1(t)\\ \alpha\xi_2(t)\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \alpha\xi_1(t)\\ 0\end{pmatrix}$, then Eq. (10) is equivalent to

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&\mathrm{I}&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0\\ \xi_2(t)\\ \xi_1(t)\\ 0\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&\frac{tI}{1+t^2}&0&0\\0&0&-\frac{\mathrm{d}}{\mathrm{d}t}&0\\0&\frac{tI}{1+t^2}&0&-\frac{\mathrm{d}}{\mathrm{d}t}\end{pmatrix}\cdot$$

$\begin{pmatrix}0\\ 0\\ -\alpha\xi_1(t)\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \alpha\xi_1'(t)\\ 0\end{pmatrix}$, which means that $\xi_1(t)\equiv 0$. Therefore the problem (9) – (10) has the unique solution $\tilde{x}(t)\equiv(0,0)^{\mathrm{T}}$, $\tilde{u}(t)\equiv 0$, and then Eqs. (11) and (12) hold, which implies that the problem (9) – (12) has the unique solution $\tilde{x}(t)\equiv(0, 0)^{\mathrm{T}}$, $\tilde{u}(t)\equiv 0$, then $\tilde{u}(t)\equiv 0$ is the optimal control for Eqs. (2) – (4).

In addition, through some numerical simulations, we can test the correctness of the obtained results. From Eqs. (2) – (4), $\xi_2(t)\equiv 0$, $\tilde{u}(t)\equiv 0$ and then the associated quadratic cost functional can be expressed as $J=\frac{1}{2}\alpha\xi_1^2(T)+\frac{1}{2}\int_0^T\xi_1^2(t)\,\mathrm{d}t$, where $\xi_1(t)$ is any function about t. Because of the arbitrariness of $\xi_1(t)$, $T=10$ and three special functions $t^{1/3}$, 2^t, t^3-3 can be chosen without loss of generality.

Obviously, the area of blue part in Fig. 2 is $\int_0^T\xi_1^2(t)\,\mathrm{d}t$ and J equals $13.925+2.3208\alpha$, $378\,190+524\,290\alpha$, $706\,830+497\,000\alpha$, respectively. Further, we can see that for any function $\xi_1(t)$, $J\geqslant 0$ since the arbitrariness of $\xi_1(t)$. Thus, the minimum $J=0$ makes $\xi_1(t)\equiv 0$. Therefore, the optimal control and relevant solution $\tilde{u}(t)\equiv 0, \tilde{x}(t)\equiv(0, 0)^{\mathrm{T}}$ can be obtained.

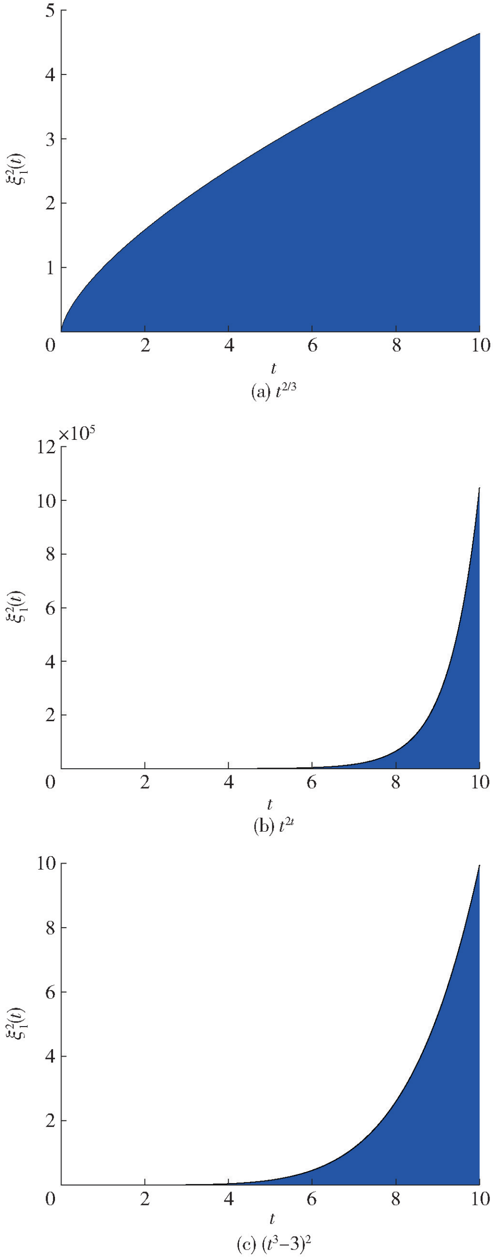


Fig. 2 Images of $\boldsymbol{\xi}_1^2(t)$ by three special functions in Example 2

Examples 1 and 2 discuss finite sum cases, and the following example will extend to infinite ones.

Example 3 Let $X = Y = L^2[0,T] \oplus L^2[0,T] \oplus \ldots$ be the orthogonal sum of an infinite number of $L^2[0,T]$, and $Z = L^2[0,T]$, $\boldsymbol{A}(t) \equiv \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}$,

$\boldsymbol{C}(t) \equiv \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \end{pmatrix}$, $\boldsymbol{B}(t) \equiv (\boldsymbol{I}, \boldsymbol{0}^T, \boldsymbol{0}^T, \ldots)^{\mathrm{T}}$, $\boldsymbol{W}(t) \equiv \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \end{pmatrix}$, $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \end{pmatrix}$, $\boldsymbol{R}(t) \equiv \boldsymbol{I}$, $\boldsymbol{S}(t) \equiv (\boldsymbol{0}, \boldsymbol{0}, \ldots)^{\mathrm{T}}$. In this example, the dotted line "..." in matrices represents an infinite number of zero matrices unless otherwise specified. And $\boldsymbol{y}_0 \equiv (\boldsymbol{I}/2, \boldsymbol{I}/3, \ldots)^{\mathrm{T}}$ is an infinite matrix whose ith entry is $\boldsymbol{I}/(i+1)$. Obviously, $K(\boldsymbol{A}(t)) = \{(\boldsymbol{f}_1^{\mathrm{T}}, \boldsymbol{0}, \boldsymbol{0}, \ldots)^{\mathrm{T}} : \boldsymbol{f}_1 \in L^2[0,T]\}$, $I(\boldsymbol{A}(t)) = X$, $\boldsymbol{P}(t) \equiv \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \end{pmatrix}$, $\boldsymbol{Q}(t) = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \end{pmatrix}$. Further,

$\boldsymbol{B}_2 \boldsymbol{R}^{-1}(t) \boldsymbol{S}_2^* - \boldsymbol{C}_{22} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \end{pmatrix}$ as an operator on the space $I(\boldsymbol{A}(t))^{\perp}$ can be regarded as being invertible, and $\boldsymbol{E}(t) = \boldsymbol{W}(t)$ is positive semi-definite. Thus we can get the optimal control for Eqs. (2) – (4) from Theorem 3.

In fact, the system (9) – (12) is equivalent to

$$\left(\begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix} \boldsymbol{x}(t) \right)' = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{0} \\ \vdots \end{pmatrix} \boldsymbol{u}(t) \tag{29}$$

$$\begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix} \boldsymbol{x}(0) = \boldsymbol{y}_0 \tag{30}$$

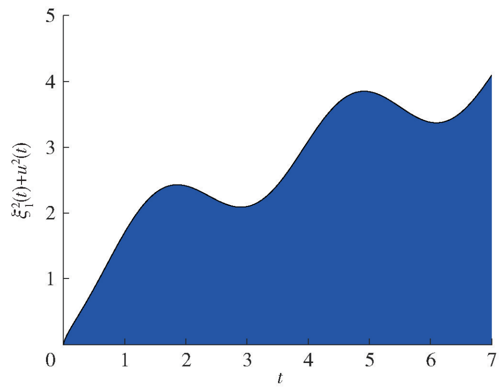
$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \ldots \\ \boldsymbol{0} & \boldsymbol{0} & \ldots \\ \vdots & \vdots & \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \end{pmatrix} \boldsymbol{u}(t) = \boldsymbol{0} \tag{31}$$

$$(\boldsymbol{0}\ \boldsymbol{0}\ \ldots) \boldsymbol{x}(t) + \boldsymbol{u}(t) = 0 \tag{32}$$

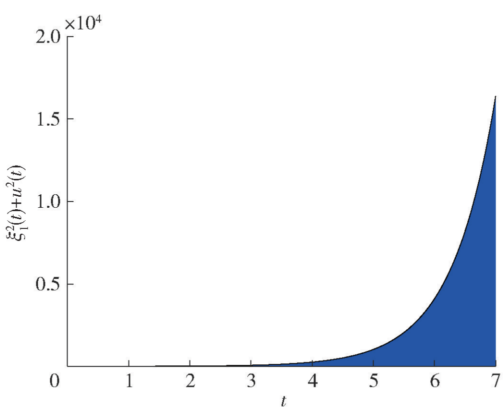
Eqs. (31) and (32) immediately lead to $\boldsymbol{\xi}_1(t) = \boldsymbol{u}(t) \equiv \boldsymbol{0}$, where $\boldsymbol{x}(t) = (\boldsymbol{\xi}_1(t)^{\mathrm{T}}, \boldsymbol{\xi}_2(t)^{\mathrm{T}}, \ldots)^{\mathrm{T}} \in X$ is the infinite matrix whose ith entry is $\boldsymbol{\xi}_i(t)^{\mathrm{T}}$. From Eq. (29), we derive that $\boldsymbol{\xi}'_i(t) = \boldsymbol{0}$, $i \geqslant 2$, which together with the initial condition (30) gives $\boldsymbol{\xi}_i(t) \equiv \boldsymbol{I}/i$, $i \geqslant 2$. Therefore the problem (9) – (12) has the unique solution $\tilde{\boldsymbol{x}}(t) \equiv (\boldsymbol{0}, \boldsymbol{I}/2, \boldsymbol{I}/3, \ldots)^{\mathrm{T}}$, $\tilde{\boldsymbol{u}}(t) \equiv \boldsymbol{0}$, here $\tilde{\boldsymbol{x}}(t)$ is the infinite matrix whose ith entry is $\boldsymbol{I}/i$ for $i > 0$, and then $\tilde{\boldsymbol{u}}(t) \equiv \boldsymbol{0}$ is the optimal control for Eqs. (2) – (4).

In addition, we can use some image simulations to examine the validity of the obtained results. For Eqs. (2) − (3), we can get $\boldsymbol{\xi}_2(0) \equiv \boldsymbol{I}/2$, $\boldsymbol{\xi}_i(t) \equiv \boldsymbol{I}/i$, $i \geqslant 3$, and then the associated quadratic cost functional can be expressed as $J = \frac{1}{2}\int_0^T (\boldsymbol{\xi}_1^2(t) + \boldsymbol{u}^2(t))\mathrm{d}t$, where $\boldsymbol{\xi}_1(t)$ and $\boldsymbol{u}(t)$ are any function about t. Because of the arbitrariness of $\boldsymbol{\xi}_1(t)$ and $\boldsymbol{u}(t)$, $T = 7$ and some special functions $\boldsymbol{\xi}_1(t) = t^{1/3}$, 2^t, $t^3 - 3$, $\boldsymbol{u}(t) = \sin t$, $\cos t$, $t^2 + 1$ can be chosen without loss of generality.

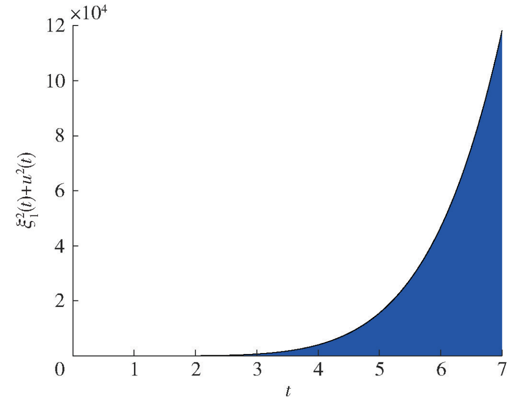
Obviously, the area of blue part in Fig. 3 is $\int_0^T (\boldsymbol{\xi}_1^2(t) + \boldsymbol{u}^2(t))\mathrm{d}t$ and J equals 9.310 7, 5 910.8, 58 854, respectively. Further, for any $\boldsymbol{\xi}_1(t)$ and $\boldsymbol{u}(t)$, $J \geqslant 0$ since the arbitrariness of $\boldsymbol{\xi}_1(t)$ and $\boldsymbol{u}(t)$. Thus, from the minimum $J = 0$, $\boldsymbol{\xi}_1(t) = \boldsymbol{u}(t) \equiv \boldsymbol{0}$ can be obtained, and then $\boldsymbol{\xi}'_2(t) = \boldsymbol{0}$, $\boldsymbol{\xi}_2(t) \equiv \boldsymbol{I}/2$. Therefore, the optimal control and relevant solution $\tilde{\boldsymbol{u}}(t) \equiv \boldsymbol{0}$, $\tilde{\boldsymbol{x}}(t) \equiv (\boldsymbol{0}, \boldsymbol{I}/2, \boldsymbol{I}/3, \ldots)^{\mathrm{T}}$ can be obtained, here $\tilde{\boldsymbol{x}}(t)$ is the infinite matrix whose ith entry is $\boldsymbol{I}/i$ for $i > 0$.



(a) $t^{2/3}+\sin^2 t$



(b) $2^{2t}+\cos^2 t$



(c) $(t^3-3)^2+(t^2+1)^2$

Fig. 3 Images of $\boldsymbol{\xi}_1^2(t) + \boldsymbol{u}^2(t)$ by three special functions

In Example 3, the range inclusion and invertibility conditions are still not necessary.

6 Conclusions

In this paper, the LQOCP for time-varying descriptor systems in a real Hilbert space is discussed. Some new sufficient conditions for the solvability of such problem are obtained by means of the Moore-Penrose inverse and space decomposition, which are important analysis tools to discuss the LQOCP for time-varying descriptor systems. The generalized inverse theory and space decomposition technique proposed in this paper is a new, useful and effective method to deal with the solvability for the LQOCP for time-varying descriptor systems, which will help gain a deeper understanding for the LQOCP.

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