Construction of compressed sensing matrixes based on the singular pseudo-symplectic space over finite fields

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Abstract

Compressed sensing (CS) provides a new approach to acquire data as a sampling technique and makes it sure that a sparse signal can be reconstructed from few measurements. The construction of compressed matrixes is a central problem in compressed sensing. This paper provides a construction of deterministic CS matrixes, which are also disjunct and inclusive matrixes, from singular pseudo-symplectic space over finite fields of characteristic 2. Our construction is superior to DeVore’s construction under some conditions and can be used to reconstruct sparse signals through an efficient algorithm.

Keywords compressed sensing matrix, singular pseudo-symplectic space, sparse signal, disjunct matrix, inclusive matrix

1 Introduction

CS is a signal processing technique for efficiently acquiring and reconstructing a signal, by finding solutions to underdetermined linear systems. Let $x \in \mathbb{R}^t$ be a discrete signal. If $x$ can be measured by $s \in \mathbb{R}$ liner projections, an $s \times t$ matrix $\Phi$ consists of the $s$ liner projections is said to be a CS matrix. If $x$ has $k \in \mathbb{R}$ nonzero entries at most, $x$ is called a $k$ sparse vector. When we obtain $y$, how to recover the original signal $x$ from $y = \Phi x$? Donoho et al. [1–2] had made it possible to obtain the major information of an original signal by seeking the sparse solution of the liner equation $y = \Phi x$.

When a matrix is given, how to judge whether it is good for CS? Candès [3] introduced a widely accepted standard of judgment: restrict isometry property (RIP).

Let $\Phi$ be an $s \times t$ matrix. For any $k$ sparse signal $x \in \mathbb{R}^t$, if there is a constant $\delta_k$ satisfying $0 \leq \delta_k < 1$ such that

$$(1 - \delta_k) \|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_k) \|x\|_2^2$$

then $\Phi$ is said to have the RIP of order $k$ and the minimum nonnegative integer $\delta_k$ is called restrict isometry constant of order $k$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be column vectors of $\Phi$. The coherence of $\Phi$, denoted by $\mu(\Phi)$, is defined as following:

$$\mu(\Phi) = \max_{i \neq j} \frac{|\langle \alpha_i, \alpha_j \rangle|}{\|\alpha_i\| \cdot \|\alpha_j\|}$$

(2)

Lemma 1 If $\Phi$ has unit-norm and coherence $\mu = \mu(\Phi)$, then $\Phi$ satisfies the RIP of order $k$ with

$$\delta_k \leq (k - 1)\mu \quad \text{for all} \quad k < 1 + (1/\mu) [4].$$

The construction of CS matrixes acts an important role in CS. Random sensing matrixes are good in theoretic and experimental, but also have many disadvantages. In order to overcome the disadvantages, DeVore [5] introduced deterministic matrixes. Recently, Li et al. [6] gave a highly efficient algorithm by using the theory of pooling design.

A pooling design can be denoted by a matrix defined on $\{0, 1\}$ whose columns and rows are indexed by items and pools respectively. When the $i$th pool contains the $j$th item, the entry at cell $(i, j)$ is 1 and 0, otherwise. A binary matrix $M$ is $(d, z)$ disjunct if for any $d + 1$ columns $C_0', C_1', \ldots, C_d'$, denoted the set of elements of...
\[ C_i', i = 0,1,\ldots,d \] by \( C_i \)
\[
\left| C_0 \setminus \bigcup_{i=1}^{h} C_i \right| \geq z \tag{3}
\]
i.e. the number of element 1 in the set \( C_0 \setminus \bigcup_{i=1}^{h} C_i \) is \( z \) at least, and is \((h,w)\) inclusive if for any \( h+1 \) columns \( C_{i_0}', C_{i_1}', \ldots, C_{i_h}' \), denoted the set of elements of \( C_i', i = 0,1,\ldots,h \) by \( C_i \)
\[
\left| C_0 \cap \left( \bigcup_{i=1}^{h} C_i \right) \right| \leq w; \quad C_i', i = 0,1,\ldots,h
\tag{4}
\]
i.e. the number of element 1 in the set \( C_0 \cap \left( \bigcup_{i=1}^{h} C_i \right) \) is \( w \) at most.


In this section, we will introduce some relevant concepts and theorems of pseudo-symplectic geometry over finite fields of characteristic 2 [16].

\[ S_1 = \begin{pmatrix} 0 & I^{(v)} \\ I^{(v)} & 0 \end{pmatrix} \]
\[ S_2 = \begin{pmatrix} 0 & I^{(v)} \\ I^{(v)} & 0 \end{pmatrix} \]
\[ S_{s,j} = \begin{pmatrix} S_\delta & 0 \\ 0 & 0 \end{pmatrix} \]

where \( \delta = 1 \) or 2. The set of all \((2v+\delta+1)\times(2v+\delta+1)\) nonsingular matrices \( T \) over \( F_q \) satisfying \( T S_{s,j} T^T = S_{s,j} \), forms a group with respect to matrix multiplication, called the singular pseudo-symplectic group of degree \( 2v+\delta+1 \) and rank \( 2v+\delta \) over \( F_q \) and denoted by \( P_{2v+\delta+1,2v+\delta}(F_q) \).

The singular pseudo-symplectic group \( P_{2v+\delta+1,2v+\delta}(F_q) \) has an action on \( F_q(2v+\delta+1)\times P_{2v+\delta+1,2v+\delta}(F_q) \rightarrow F_q(2v+\delta+1) \),
\[
((x_1,x_2,\ldots,x_{2v+\delta+1}), T) \mapsto (x_1,x_2,\ldots,x_{2v+\delta+1})T
\]

The vector space \( F_q(2v+\delta+1) \) together with this group action is called the \((2v+\delta+1)\) dimensional singular pseudo-symplectic space over the finite field \( F_q \) of characteristic 2. Let \( E \) be the subspace of \( F_q(2v+\delta+1) \) generated by \( e_{2v+\delta+1}, e_{2v+\delta+2},\ldots, e_{2v+\delta+1} \). An \( m \) dimensional subspace \( P \) of \( F_q(2v+\delta+1) \) is called a subspace of type \((m,2s+\tau,s,e,k)\) if

1) \( P \) is a subspace of type \((m,2s+\tau,s,e,k)\), see Ref. [16].

2) \( \dim(P \cap E) = k \).

We use \( P' \) to denote the matrix which represents the subspace \( P \). Denote the set of subspaces of type \((m,2s+\tau,s,e,k)\) in \( F_q(2v+\delta+1) \) by \( \mathcal{M}(m,2s+\tau,s,e,k,2v+\delta+1,2v+\delta) \), where \( \delta = 1 \) or 2, \( \tau = 0,1 \) or 2, and \( e = 0 \) or 1. Let \( N(m,2s+\tau,s,e,k,2v+\delta+1,2v+\delta) = |\mathcal{M}(m,2s+\tau,s,e,k,2v+\delta+1,2v+\delta)| \).

**Lemma 2** \( \mathcal{M}(m,2s+\tau,s,e,k,2v+\delta+1,2v+\delta) \) is non-empty if and only if
\[(\tau, e) = \begin{cases} (0, 0), (1, 0), (1, 1), \text{ or } (2, 0); & \delta = 1, \\ (0, 0), (0, 1), (1, 0), (2, 0), \text{ or } (2, 1); & \delta = 2, \end{cases} \]

\[
\text{max} \left\{ 0, m - v - s - \left[ (\tau + \delta - 1)/2 \right] - e \right\} \leq k \leq \text{min} \left\{ l, m - 2s - \text{max} \left\{ \tau, e \right\} \right\}
\]

hold simultaneously.

Moreover, if \( \mathcal{N}(m, 2s + r, s, e, k; 2v + \delta + l, 2v + \delta) \neq \emptyset \), the value of \( N(m, 2s + r, s, e, k; 2v + \delta + l, 2v + \delta) \) can be found in Ref. [16].

Let \( P \) be an \( m \) dimensional vector subspace of \( F_q^{(2v + \delta + l)} \). Denote by \( P^\perp \) the set of vectors which are orthogonal to every vector of \( P \), i.e. \( P^\perp = \{ y \in F_q^{(2v + \delta + l)} \mid y^T x = 0 \} \) for all \( x \in P \). Obvioulsy, \( P^\perp \) is a \( (2v - m + \delta + l) \)-dimensional subspace of \( F_q^{(2v + \delta + l)} \) and is called the dual subspace of \( P \).

\[
P_{0i}^\perp = \begin{pmatrix}
m_{0i} & v - (m_{0i} - k_i) & m_{0i} - k_i & v - (m_{0i} - k_i) & 2 & k_i \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}^{T} \begin{pmatrix}
m_{0i} - k_i \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

Denote by \( G \) the set of all subspaces of type \( (m_i, 0, 0, 1, k) \) containing \( P_0 \) in \( P^\perp_0 \). \( G = \{ P_1, P_2, \ldots, P_t \} \), where \( P_i, i = 1, 2, \ldots, t \) are different subspaces of type \( (m_i, 0, 0, 1, k) \) and \( t = |G| \). Denote the set of all subspaces of type \( (m_i, 0, 0, 1, k_2) \) containing \( P_0 \) in \( P^\perp_0 \) by \( H \). \( H = \{ Q_1, Q_2, \ldots, Q_s \} \), where \( Q_j, j = 1, 2, \ldots, s \) are different subspaces of type \( (m, 0, 0, 1, k_2) \) and \( s = |H| \).

Let \( \Phi_0 = (a_{ij})_{s \times 2} \) be a matrix defined on \( \{0, 1\} \), where

\[
a_{ij} = \begin{cases} 1; & Q_j \subseteq P_i \\ 0; & \text{otherwise} \end{cases}
\]

Let \( R \) be a subspace of type \( (m, 0, 0, 1, k) \) containing \( P_0 \) in \( P^\perp_0 \), then \( R \) has the following matrix representation

\[
R' = \begin{pmatrix}
I_{(m_{0i} - k_i)} & 0 & 0 & 0 & 0 & R_0 \\
0 & 0 & R_2 & 0 & 0 & 0 \\
0 & 0 & R_3 & 0 & 0 & 0 \\
I^{(k_i)} & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

We can obtain that \( (R^T_1 R^T_2 R^T_3 R^T_1) \) is a subspace matrix representation of type \( (m - m_0, 0, 0, 1, k_i - k) \) in \( F_q^{(2v - m_0 + k_0 + 2l - k - k_i)} \) and

\[
t = N(m - m_0, 0, 0, 1, k_i - k) - 2(v - m_0 + k_0 + 2l - k_0) + 2(v - m_0 + k_0) + 2)
\]

\[
\prod_{j=1}^{v - m_0 + k_0} \frac{(q^{2j} - 1)^{l_j - k_j}}{(q^{j} - 1) \prod_{i=1}^{l_j(2j - 1)} (q^{j} - 1)}
\]

3 The construction

Let \( m, m_0, i, k_0, k_1, k_2, \) be integers satisfying \( 0 \leq m_0 < m \leq v \), \( 0 < k_0 \leq k_0 \leq l \). Let \( P_0 \) be a fixed subspace of type \( (m_0, 0, 0, 0, k_0) \) in \( F_q^{(2v + \delta + l)} \), then \( P_0^\perp \) is a subspace of type \( (2v - m_0 + k_0 + 2l, 2v - m_0 + k_0 + 2, v - m_0 + k_0, 1, l) \). Let the matrix representation of \( P_0 \) be

\[
\begin{pmatrix}
I_{(m_0 - k_0)} & 0 & 0 & 0 & 0 & v - (m_0 - k_0) \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Similarly,
\[ s = N(m_1 - m_0, 0, 1, k_2 - k_0; 2(v - m_0 + k_0) + 2 + l - k_0, 2(v - m_0 + k_0) + 2) = \prod_{i=1}^{2} \frac{(q^{l-k_0} - 1) \prod_{j=1}^{2} (q^{l-k_0} - 1)}{q(m_1 - m_0 + k_0 + 2)} \]  \hspace{1cm} (7)

Let \( L \) be the number of 1 in every column, i.e. the number of subspaces of type \((m_1, 0, 0, 1, k_1)\) contained in a subspace of type \((m, 0, 0, 1, k_1)\). From Ref. [16] we know \( L = N(m_1 - m_0, 0, 1, 1 - k_0; m_1 - m_0, 0, 0, 1, 1 - k_0; 2(v - m_0 + k_0) + 2) \), then \( k_0, \nu - m_0 + k_0) = N(m_1 - m_0 - 1, 0, 1 - k_0; 2(v - m_0 + k_0) + l - \sum_{i=m_1-m_0-k_0}^{2} (q^{l-k_0} - 1) \prod_{j=1}^{2} (q^{l-k_0} - 1)) \).

From Lemma 1, let \( \Phi = (1/\sqrt{L}) \Phi \), then \( \mu(\Phi) = \mu(\Phi) \). Therefore, the matrix \( \Phi \) satisfies RIP with \( \delta_k \leq (k-1)\mu \), for any \( k < 1/\mu + 1 \approx q^{m_1-m_0+1-k_0} + 1 \). Thus, \( k_{max} \approx q^{m_1-m_0-1-k_0} \).

Let us compare the matrix which we have constructed above with the matrix constructed by Devore. Let \( \Phi \) be the \( s \times t \) matrix, in DeVore’s construction [5], with \( s_i = q_i^2 \), \( t_i = q_i^{t_i} \) and \( k_{max} = q_i/r \), where \( 1 < r < q_i \) and \( q_i \) is a prime power.

Let \( m_0 = 2 \), \( m_1 = v - 1 \), \( k_1 = 1 \), \( k_2 = l \) and \( q = q_i \), then \( s = q_i \approx q_i^{v-1} \), \( t_i = q_i^{t_i} \), \( k_{max} = q_i^{t_i} \).

When \( q \) is large enough, we have \( s = q_i^{v-1} \), \( t \approx q_i^{v-1} \), \( k_{max} = q_i^{t_i} \).

Let \( s/t = s_i/t_i \), then \( r = 3l - v - 4 \).

Furthermore, let \( v - l = 4 \) and \( l \geq 2 \). We can obtain \( k_{max} = \frac{q}{2l} < k_{max} = q \).

Therefore, our construction is superior to DeVore’s construction under some conditions.

Then we will discuss the disjunct and inclusive property of matrix \( \Phi \).

Theorem 1 \( \Phi \) is a \((d, z)\) disjunct matrix, where \( z = N(m_1 - m_0 - d - 1, 0, k_2 - k_0; m_1 - m_0 - d - 1, 0, k_2 - k_0; 2(v - m_0 + k_0) + l - k_0, \nu - m_0 + k_0) \) and \( d \leq m_1 - m_0 \).

Proof Let \( C_0, C_1, ..., C_{d} \) be \( d+1 \) distinct columns of \( \Phi \), i.e. different subspaces of type \((m_0, 0, 1, k_1)\) containing \( P_0 \) in \( P_0 \) denoted by \( C_i, i = 0, ..., d \). There must exist a subspace of type \((1, 0, 0, 0)\) or \((1, 0, 0, 1)\) denoted by \( D_j \), where \( 1 < i < d \), which is not contained in \( P_0 \), satisfying \( D_j \subseteq C_0 / C_i \). Let \( A = D_i \oplus D_j \oplus ... \oplus D_{d} \oplus P_0 \). As it may exist \( D_j \) \( (i \leq j < d) \), \( A \) is a subspace of type \((v, 0, 0, 1, k_2')\),
where \( y \leq d + m_0 \), \( k_s \leq k'_2 \leq k_2 \). It is easy to find a subspace of type \( (m_i - y, 0, 0, 1, k_i - k'_2) \) from the complementary space of \( A \) in \( C_0 \), denoted by \( A' \). Then \( A \oplus A' \) is a subspace of type \( (m_i, 0, 0, 1, k_i) \) which contains \( P_0 \) and \( A \oplus A' \subseteq C_0 \setminus C_i \), \( 1 \leq i \leq d \).

The number of \( A' \) equals to the number of subspaces of type \( (m_i - m_0 - d, 0, 0, 1, k_i - k_0) \) contained in a subspace of type \( (m_i - m_0 - d, 0, 0, 1, k_i - k_0) \), i.e. \( N(m_i - m_0 - d, 0, 0, 1, k_i - k_0; m - m_0 - d, 0, 1, k_i - k_0; 2(v - m_0 + k_0) + 2 + l - k_0, v - m_0 + k_0) \), which is also the number of \( A \).

When \( M(m_i, 0, 0, 1, k_i - x; m, 0, 0, 1, k_i - x; 2(v - m_0 + k_0) + 2 + l - k_0, v - m_0 + k_0) \) is non-empty, let \( \Phi \) be a subspace of type \( (m_i, 0, 0, 1, k_i) \), \( \Phi \) is a subspace of type \( (m_i, 0, 0, 1, k_i) \) containing \( \Phi \) and \( \Phi \). Then, \( \Phi \) is a subsequence \( \Phi \) or \( \Phi \) and \( \Phi \).

Next we will find the minimal \( P \) containing \( \Phi \) contains \( \Phi \) containing \( \Phi \) or \( \Phi \). Then \( \Phi \).

Proof Let \( C'_0, C'_1, C'_2, \ldots, C'_{d} \) be \( d + 1 \) distinct columns of \( \Phi \), i.e. different subspaces of type \((m, 0, 0, 1, k_i)\) denoted by \( C_i \), \( i = 0, 1, \ldots, d \). The number of \( 1 \) in the column indexed by \( C_0 \) equals to \( L \), the number of subspaces of type \((m_i, 0, 0, 1, k_i)\) containing \( P_0 \) in \( C_0 \). Denote the number of subspaces of type \((m_i, 0, 0, 1, k_i)\) containing \( P_0 \) in \( C_0 \) but not in \( C_i \), \( 1 \leq i \leq d \) by \( |C_0 \setminus (C_0 \cup \cdots \cup C_d)| \), which equals to \( |C_0 \setminus (C_0 \cap C_1) \cup \cdots \cup (C_0 \cap C_d)| \). Next we will find the minimal value of \( |C_0 \setminus (C_0 \cap C_1) \cup \cdots \cup (C_0 \cap C_d)| \).

We can assume that \( C_i \cap C_0 \) is a subspace of type \((m - 1, 0, 0, 1, k_i)\) or \((m - 1, 0, 0, 1, k_i - 1)\) containing \( P_0 \). The number of subspaces of type \((m_i, 0, 0, 1, k_i)\) contained in the subspace of type \((m - 1, 0, 0, 1, k_i)\) or \((m - 1, 0, 0, 1, k_i - 1)\) is \( L_i \) or \( L_{(i)} \). And for \( C_j \) and \( C_j \), \( 1 \leq i, j \leq d \), \( C_i \cap C_j \) is subspace of type \((m - 2, 0, 0, 1, k_i)\), \((m - 2, 0, 0, 1, k_i - 1)\) or \((m - 2, 0, 0, 1, k_i - 2)\) which contains \( P_0 \). The number of subspaces of type \((m_i, 0, 0, 1, k_i)\) contained in the above subspace is \( L_{j} \), \( L_{j} \) or \( L_{j} \). Let \( L_{j} = \max \{L_{j - 1}, L_{j} - L_{j - 1}, L_{j} - L_{j - 1}, L_{j} - L_{j - 1}\} \). The number of subspaces of type \((m_i, 0, 0, 1, k_i)\) contained in \( C_0 \) but in \( C_i \), \( 1 \leq i \leq d \) is \( z \).

\[ z = L - d \max \{L_{1}, L_{2}\} + (d - 1) \min \{L_{1}, L_{2}\} = L - \max \{L_{1}, L_{2}\} - (d - 1) \max \{L_{1}, L_{2}\} - \min \{L_{1}, L_{2}\} \]

**Theorem 2**

Let \( m - m_0 \geq 3 \), \( b = (L - \max \{L_{1}, L_{2}\})/L_{6} \), where
\[ L - \max \{ L_1, L_2 \} - (d - 1) L_b \]  
(22)

If \( \Phi \) is a \((d,z)\) disjunct matrix with \( z > 0 \), then \( d < (L - \max \{ L_1, L_2 \})/L_b + 1 \). Let \( b = (L - \max \{ L_1, L_2 \})/L_b \), then \( 1 \leq d \leq b \).

**Corollary 1** Let \( m - r \geq 3 \), \( k_0 = k_2 \), \( b = (q^{-m_1-1-k_2}\rightarrow \rightarrow q)/(q^{-m_1-1-k_2} \rightarrow \rightarrow 1) \), then \( 1 \leq d \leq b \), \( \Phi \) is a \((d,z)\) disjunct matrix, where \( z = L - dL_3 - (d - 1)L_3 \).

**Proof** As

\[
L_1 = \left( q^{m_1-1-k_2} - q^{-1} \right) / \left( q^{m_1-1-k_2} - q^{-1} \right)
\]

(23)

\[
L_4 = \left( q^{m_1-1-k_2} - q^{-1} \right) / \left( q^{m_1-1-k_2} - q^{-1} \right)
\]

when \( k_2 = k_0 \), we get \( L_1 < L_3 \) and \( L_4 > L_3 \). Since \( L_1 > L_2 \), we get

\[
z = L - dL_3 - (d - 1)L_3 = \sum_{i=1}^{m_1-1-k_2+1} (q^{i-1}) / \left( q^{m_1-1-k_2+1} - q^{i-1} \right)
\]

(24)

where

\[
k = \left( q^{m_1-1-k_2} - q^{i-1} \right) / \left( q^{m_1-1-k_2} - q^{i-1} \right)
\]

As \( z > 0 \), we have \( d < (q^{m_1-1-k_2} \rightarrow \rightarrow 1 ) / (q^{m_1-1-k_2} \rightarrow \rightarrow 1) + 1 \). Let \( b = (q^{m_1-1-k_2} \rightarrow \rightarrow q)/(q^{m_1-1-k_2} \rightarrow \rightarrow 1) \), then \( 1 \leq d \leq b \).

**Theorem 3** Let \( m - m_1 - k_1 \geq 2 \), \( k_0 = k_2 \), \( 1 \leq d \leq b \), then \( \Phi \) is not a \((d,z+1)\) disjunct matrix, where \( z \) is the same as appeared in Corollary 1.

**Proof** It is sufficient to prove that the \( z \) mentioned in Corollary 1 can be reached. Let \( C_0 \) have the following matrix representation

\[
C_0 = \left( f^{(m_1-k_0)} 0 0 0 0 0 \right)^T
\]

(25)

Let \( L \) be the number of \((m,0,0,1,k_1)\) containing \( P_0 \) in \( C_0 \). \( E^T \) has the following form

\[
E = \left( f^{(m_1-k_0)} 0 0 0 0 0 \right)^T
\]

(26)

Let \( F \) be a fixed subspace of type \((m-1,0,0,1,k_1)\) containing \( E \) in \( C_0 \). Then the number of \( F \) is \( q^{(m_1-k_0) \rightarrow \rightarrow 1} \). When \( m - m_0 - k_1 - k_0 \geq 2 \), we have \( q^{(m_1-k_0) \rightarrow \rightarrow 1} < q^{(m_1-k_0) \rightarrow \rightarrow 1} \). Hence \( 1 \leq d \leq b \). For \( 1 \leq d \leq b \), we can choose \( d \) distinct subspaces of type \((m-1,0,0,1,k_1)\) containing \( E \) in \( C_0 \), denoted by \( F_i \) \((1 \leq i \leq d)\). For each \( F_i \) \((1 \leq i \leq d)\), we can choose a \( C_i \) s.t. \( C_i \cap C_0 = F_i \). Then for each pair of \( F_i, F_j \) \((1 \leq i, j \leq d)\), \( F_i \cap F_j = E \). The number of subspaces of type \((m_1,0,0,1,k_1)\) contained in \( F_i \) \((1 \leq i \leq d)\) and \( E \) can be reached.

**Theorem 4** Let \( k_0 = k_2 \), \( 2m_1 - k_1 - k_2 \leq m_0 + 1 \), \( m_1 - m_0 \geq 2 \). Then the maximum value of \( d \) satisfying \( 1 \leq d \leq b \) is \( q^{m_1-k_0} \).

**Proof** Firstly, when \( m_1 - m_0 \geq 2 \),
\[ b - d = q \left( q^{m-\delta_k-\delta_k-1} - 1 \right) - q^{m-\delta_k} = \frac{q^{m-\delta_k-1} - q}{q^{m-\delta_k-1} - 1} \geq 0 \] (26)

Secondly, when \( 2m_1 + k_i - k_z \leq m + m_0 + 1 \), we have
\[ q^{m-\delta_k-1} \leq q^{m_1 + k_2 - k_i} \] (27)

So
\[ q^{m-\delta_k-1} - q q^{m_1 + k_2 - k_i} - 1 < 1 \] (28)

Hence \( 0 < b - d < 1 \), i.e. the maximum value of \( d \) satisfying \( 1 \leq d \leq b \) is \( q^{m-\delta_k-1} \).

A binary matrix \( M \) is \( \delta \) disjunct if for any \( d + 1 \) columns \( C_{\delta}, C_{\delta+1}, ..., C_d \), the set of elements of \( C_{\delta} \), \( \delta = 0, 1, ..., d \), by \( C_{\delta} \), \( C_{\delta+1}, ..., C_d \) disjunct and \( (h, w) \) inclusive if for any \( h+1 \) columns \( C_{\delta}, C_{\delta+1}, ..., C_d \), denoted the set of elements of \( C_{\delta} \), \( h = 0, 1, ..., h \) by \( C_{\delta} \), \( C_{\delta+1}, ..., C_d \) disjunct and \( (w, z) \), \( w = 1, 0, 0, 1, k_2 \) contain \( p_0 \) in \( p_0^w \) respectively.

Similarly, we can prove that \( \Phi \) is a \( (k', z') \) disjunct matrix and \( (k, w) \) inclusive matrix with \( z' > w \) which means \( k' < q^n/m/2 \). So for a \( k \) sparse signal \( x \in \mathbb{R}^{(k)} \), we can process and recover the signal by \( \Phi \) when \( k < q^n/m/2 \), and if \( q^n/m/2 + 1 > k > (1/2)q^n/m - 1 \), the \( k \) sparse signal cannot be recovered.

Let \( \Phi \) be the matrix whose columns and rows are indexed by all subspaces of type \( (m, 0, 0, 1, k_1) \) and \( (m_1 + 1, 0, 0, 1, k_2) \) respectively.

**Example 1** Let \( m = 2, k_0 = 1, r = 4, k_2 = 1, m = 8, k_2 = 2, l = 3, p = 1, q = 4 \), then the logarithms of \( \Phi = (a_\delta)_{\delta=0} \) are \( s = N(2, 0, 0, 1, 0, 2, 7 + 2, 2, 7) \), \( t = N(6, 0, 0, 1, 0, 1, 2, 7 + 2, 2, 7 + 2) \), \( L = 340, \mu = 21/85, k \leq 5 \) and \( b = 4 \). So for \( 1 \leq d \leq b \), \( \Phi \) is a \( (k, z) \) disjunct matrix and \( (k, w) \) inclusive matrix with \( z = 320 - 28d \) \( w = 20 + 28d \). From \( z > w \), deduces \( d \leq 2 \). Hence for a \( k \) sparse signal \( x \in \mathbb{R}^{(k)} \) with \( k \leq 2 \), we can

1) Compress and recover the 2-sparse signal by (2,264) disjunct and (2,76) inclusive matrix.
2) Compress and recover the 1-sparse signal by (1,292) disjunct and (1,48) inclusive matrix.

For a \( k \) sparse signal \( x \in \mathbb{R}^{(k)} \) with \( 3 \leq k \leq 5 \), we need to use \( \Phi' \) whose logarithms are \( s' = N(2, 0, 0, 0, 2, 7 + 2, 7), t' = N(5, 0, 0, 1, 2, 7 + 2, 7), L' = 5712, \mu' = 1/17, k' \leq 17, \) \( \beta' = 17. \) And \( z' = 5696 - 320d', w' = 16 - 320d' \). From \( z' > w' \), deduces \( d' \leq 8 \). Hence for a \( k' \) sparse signal \( x \in \mathbb{R}^{(k)} \) with \( k' \leq 2 \), we can

1) Compress and recover the 1 sparse signals by (1, 376) disjunct and (1,336) inclusive matrix.
2) Compress and recover the 2 sparse signals by (2, 5 056) disjunct and (2,656) inclusive matrix.

3) Compress and recover the 3 sparse signals by (3,4 736) disjunct and (3,976) inclusive matrix.

4) Compress and recover the 4 sparse signals by (4, 4 416) disjunct and (4,1 296) inclusive matrix.

5) Compress and recover the 5 sparse signals by (5, 4 096) disjunct and (5,1 616) inclusive matrix.

Although for 1 sparse signals and 2 sparse signals, we can also use $\Phi'$, but $s'/s \approx 4^{4}/15$ that means it needs nearly $4^{2}$ times measurements.

4 Conclusions

For the CS matrix $\Phi$ which we have constructed and $\Phi_1$ constructed by DeVore. Suppose $\Phi$ and $\Phi_1$ have the same effectivity i.e. $s/t = s_1/t_1$, our construction has the better sparsity of sensing matrix under some position. Moreover, the deterministic CS matrix $\Phi$ has some disjunct and inclusive properties and it can be used to reconstruct sparse signals through an efficient algorithm.

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References


