Performance analysis of ARQ schemes with code combining over Nakagami-\(m\) fading channel

XU Wen-bo\(^{(x)}\), NIU Kai, LIN Jia-ru, HE Zhi-qiang

School of Information and Telecommunication Engineering, Beijing University of Posts and Telecommunications, Beijing 100876, China

Abstract

This article investigates the performance of hybrid automatic repeat request (HARQ) with code combining over the ideally interleaved Nakagami-\(m\) fading channel. Two retransmission protocols with coherent equal gain code combining are adopted, where the entire frame and several selected portions of the frame are repeated in protocols I and II, respectively. Protocol II could be viewed as a generalization of the recently proposed reliability-based HARQ. To facilitate performance analysis, an approximation of the product of two independent Nakagami-\(m\) distributed random variables is first developed. Then the approximate analysis is utilized to obtain exact frame error probability (FEP) for protocol I, and the upper bound of the FEP for protocol II. Furthermore, the throughput performance of both two protocols is presented. Simulation results show the reliability of the theoretical analysis, where protocol II outperforms protocol I in the throughput performance due to the reduced amount of transmitted information.

Keywords code combining, HARQ, Nakagami-\(m\) fading, performance analysis

1 Introduction

In wireless communications, HARQ, which combines automatic repeat request (ARQ) and forward error correction, is an effective technique to ensure the reliability of data transmission. When a received frame is detected to be in error at the receiving end, usually through the use of cyclic redundancy check (CRC), a retransmission request is feedback to the transmitter. Then retransmission will be performed till either the receiver successfully decodes or a maximum retransmission limit \(F\) is reached. In previous works, multiple retransmission strategies have been proposed. Among them, incremental redundancy (IR) ARQ, which combines information from multiple retransmissions [1], has received considerable attention.

The performance of IR ARQ and its variations with code combining has been previously evaluated over different channels. Ref. [2] utilizes multiple repeated copies with coherent equal gain code combining (EGC) over the ideally interleaved Rayleigh fading channel, and gives the analytical performance. Authors in Ref. [3] provide the performance bounds on Turbo coded IR HARQ over additive white Gaussian noise (AWGN) channel, where the entire coded frame or portions of the frame is retransmitted. To improve the throughput performance, a reliability-based hybrid ARQ (RBHARQ) scheme exploiting received frame reliability is proposed in Ref. [4], where only several selected bits instead of the entire frame are repeated. An alternative RBHARQ algorithm is designed in Ref. [5] that maximizes throughput under the maximum delay constraint. With perfect knowledge of the channel state information (CSI), the design therein specifies the size of successive retransmissions.

The studies on performance in aforementioned works are either based on event-driven simulations, or limited to the analytical model over AWGN or Rayleigh fading channels. In this article, the performance of HARQ protocols with code combining will be analyzed under more general assumptions. First, Nakagami-\(m\) fading channel will be considered, which includes Rayleigh channel as a special case. Second, the authors relax the requirement of perfect CSI in Ref. [5], and assume that only cophasing can be fulfilled at the receiving
end, which enables the adoption of coherent EGC similar to Ref. [2]. Traditionally, the performance analysis of coherent EGC over Nakagami-\(m\) fading channels is tedious, which generally requires the knowledge of the probability density function (pdf) of the sum of Nakagami-\(m\) random variables (refer to Nakagami sum later). Hence, the authors propose an approximation technique to avoid such requirement.

Two retransmission protocols are considered. Specifically, protocol I retransmits the entire frame and protocol II retransmits portions of the frame, where the latter can be viewed as a generalization of RBHARQ. To the best of the authors’ knowledge, the performance of RBHARQ has not been investigated analytically over the Nakagami-\(m\) fading channels. By extending the technique in Ref. [2], it is revealed that the product of two independent Nakagami-\(m\) distributed random variables (Nakagami product) can be approximated by the sum of two independent Gamma random variables (Gamma sum). Employing the proposed approximation, the authors give the exact frame error probability (FEP) for protocol I and the upper bound of FEP for protocol II. Afterwards, the throughput is presented based on the derived FEPs. The agreement of the simulated and analytical results well validates the reliability of the analysis. The results also show that protocol II embraces a better throughput results because less information is required to be repeated.

The remainder of this article is organized as follows. The system model is described in Sect. 2. Sect. 3 develops the performance of two HARQ protocols. Analytical and simulated results are given in Sect. 4. Sect. 5 concludes this article.

2 System description

Consider a wireless network with HARQ techniques. The source node \(S\) transmits a coded frame to the destination \(D\), where the frame errors are examined by the CRC code. If a frame passes the CRC, \(D\) sends an acknowledgement (ACK) message to request a new frame; otherwise, a negative acknowledgement (NACK) is feedback to request retransmission. In the latter case, the retransmission procedure, which will be discussed in the following, continues till either no error occurs at \(D\) or a hard limit on the maximum number of allowable transmissions per frame, \(F_1\), is met. For convenience, assume an ideal CRC and the ACK/NACK is received reliably.

Two protocols will be adopted to perform retransmission. In protocol I, the entire frame will be repeated when \(D\) fails, while only portions of the frame will be repeated in protocol II. The motivation to utilize the latter is twofold. First, the independent fading channels experienced by different segments in the same frame are expected to be exploited. Second, the throughput efficiency is expected to be improved because less information is retransmitted. Protocol II can be viewed as a general extension of RBHARQ scheme in Ref. [4], where the retransmitted segment could be selected as the one that contains the maximum number of unreliable bits at the destination.

Assume that perfect CSI is not available while the cophasing could be realized at \(D\). Then the coherent EGC will be adopted. When binary phase shift keying is used, the combining result after \(f\) \((f=1,2,...,F)\) transmissions will be

\[
y_{f,i} = \sqrt{E} \sum_{m=1}^{\infty} h_{m,i} \cdot x_{m,i} + \sum_{m=1}^{\infty} n_{m,i}
\]

where \(t (t=1,2,...,f)\) denotes different transmission attempts. \(x_{m,i} \in \{1,-1\}\) is the coded symbol of \(R\)-rate convolutional code with \(j (j=1,2,...,N)\) representing the bit number and \(N\) the frame length. \(E_s\) is the transmitted energy per coded symbol and \(n_{m,i}\) denotes the zero-mean AWGN with variance of \(N_b/2\). \(h_{m,i}\) corresponds to the Nakagami-\(m\) fading envelope of the \(j\)th bit at the \(i\)th transmission attempt with \(E(h_{m,i}^2) = \Omega\), where \(E()\) is the statistical average operator. Suppose that interleaving technology is employed to randomize the burst errors, and the fading amplitudes at different time could be assumed to be independent and identically distributed Nakagami-\(m\) variables.

When convolutional code is used, the upper bound on FEP can be expressed by [6]

\[
P_f \leq 1 - \left(1 - \sum_{d_e} a(d) P_2(d)\right)^B
\]

where \(B\) is the number of trellis branches in a codeword, \(a(d)\) denotes the number of error events with Hamming weight \(d\), and \(d_e\) means the free distance of the convolutional code. Without loss of generality, assume that all zero information bits are used, and then \(P_2(d)\) stands for the pairwise error probability (PEP) of incorrectly decoding a codeword of weight \(d\).

3 Performance analysis

3.1 FEP of protocol I

Since the entire frame will be repeated in protocol I, the combined result is given in Eq. (1) with \(x_{f,i} = x_{i,f}\).
\(t = 2, 3, ..., f\). Denote the FEP with \(f\) transmissions as \(P_0(f)\) and the corresponding PEP as \(P_1'(d)\). According to Ref. [2], the error performance of ARQ is mostly dependent on the output of the combined frames from \(F\) successive receptions containing detected error, which implies the FEP for this system will be \(P_0(F)\).

With Viterbi algorithm at the destination, \(P_1'(d)\) can be given by

\[
P_1'(d) = E \left( \frac{2F}{\beta N_0} \sum_{i=1}^{\infty} \sum_{j=1}^{l_i} h_{i,j}^2 \right) \tag{3}
\]

Traditionally, one has to determine the pdf of Nakagami sum to obtain the expectation in Eq. (3). However, it is tedious to perform such analytical investigation, which may explain the reason that few works are devoted to the performance analysis of coherent EGC. In the following sections, the authors propose an approximation technique on the Nakagami product to simplify the analysis.

While Ref. [2] points out that the Rayleigh product can be extended to more general case, i.e.,

\[
v_0 v_i \approx \frac{E(v_0 v_i)}{E(v_0^2 + v_i^2)} (v_0^2 + v_i^2) \tag{4}
\]

where \(v_0\) and \(v_i\) are both Nakagami-\(m\) random variables with the pdf being \(f_{v_i}(x) = (2m^m x^{2m-1})/[\Gamma(m)v_i^m] \exp(-mx/V_i)\) \((E(v_i^2) = V_i, i = 0, 1)\). It is easy to obtain

\[
A = \frac{E(v_0 v_i)}{E(v_0^2 + v_i^2)} = \frac{\Gamma^2 (m + \frac{1}{2}) \sqrt{V_0 V_i}}{\Gamma(m) \Gamma(m + 1)(V_0 + V_i)} \tag{5}
\]

where the Gamma function is defined by \(\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx\).

The reasonability of Eq. (4) can be verified by the pdf tightness as that in Ref. [2]. First, the pdf of \(\phi = v_0 v_i\) is given by

\[
f_{\phi}(x) = \int_{-\infty}^{\infty} f_{v_0}(z) f_{v_i}(x/z) \frac{2m^{2m}}{V_0^m V_i^{m} \Gamma^2(m)} x^{2m-1} e^{-x} dx \tag{6}
\]

where the last equation can be found in Ref. [7] with \(K_j()\) being the zero-order modified Bessel function. Furthermore, because the square of a Nakagami-\(m\) variable follows a Gamma distribution and the pdf of a Gamma variable \(v_j^2\) is \(f_{v_j^2}(x) = (m^m x^{m-1})/[\Gamma(m)\Gamma^2(m)] \exp(-mx/V_j)\) \((i = 0, 1)\) [8], the pdf of \(\phi = A(v_0^2 + v_i^2)\) will be

\[
f_{\phi}(x) = \frac{1}{A} \int_0^{x/A} f_{v_i^2}(z) f_{v_0^2} \left( \frac{x}{A} - z \right) \frac{m^{2m}}{\Gamma_0^m V_i^m \Gamma^2(m)} e^{-\frac{m}{V_i}z} dz \tag{7}
\]

where the last equation is learnt from Ref. [7] with \(\phi = A(v_0^2 + v_i^2)\) being the Degenerate Hypergeometric function.

Based on Eqs. (6) and (7), Fig. 1 presents the pdfs of both \(\phi\) and \(\phi\) at different values of \(m\) and \(V_0\), and Table 1 shows the corresponding mean square error (MSE) results between the exact and approximated pdfs. It is observed that the pdf in Eq. (7) offers a reasonable approximation to the exact pdf in Eq. (6), and the curves matches better when \(m\) is larger.

![Fig. 1 Comparison between the exact and approximated pdfs of Nakagami product](image)

**Table 1** MSE at different \(m\) and \(V_0\)

<table>
<thead>
<tr>
<th>(m)</th>
<th>(V_0)</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>1.9 \times 10^{-3}</td>
</tr>
<tr>
<td>1</td>
<td>1.0</td>
<td>2.5 \times 10^{-3}</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>1.487 \times 10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>5.737 \times 10^{-4}</td>
</tr>
</tbody>
</table>

With the verification of Eq. (4), the square of Nakagami sum in Eq. (3) will be approximated by

\[
\left( \sum_{i=1}^{\infty} \sum_{j=1}^{l_i} h_{i,j}^2 \right)^2 \approx \sum_{i=1}^{\infty} \sum_{j=1}^{l_i} h_{i,j}^2 + 2A \sum_{i=1}^{\infty} \sum_{j=1}^{l_i} (h_{i,j}^2 + h_{i,j}^2) = (1 + 2A(f - 1)) \sum_{i=1}^{\infty} \sum_{j=1}^{l_i} h_{i,j}^2 \tag{8}
\]

Now, it is ready to simplify Eq. (3), which is written by

\[
P_1'(d) \approx E \left( \frac{2F}{\beta N_0} (1 + 2A(f - 1)) \sum_{i=1}^{\infty} \sum_{j=1}^{l_i} h_{i,j}^2 \right) \tag{9}
\]
with \( A_i = \frac{\Gamma^2(m + 1/2)}{2(m)\Gamma(m + 1)} \).

Denote \( \lambda = (2E_s/f_{\text{SNR}})(1 + 2A_i(f - 1)) \) and \( x_{ij} = \lambda_{ij} \) \((t = 1,2,\ldots,f \text{ and } j = 1,2,\ldots,d)\). Using an alternative representative form of \( Q(x) \) [9], i.e.,

\[
Q(x) = \frac{1}{\pi} \int_0^{x/2} \exp \left( -\frac{x^2}{2\sin^2\theta} \right) d\theta
\]

Eq. (9) can be calculated as

\[
E \left[ \sum_{j=1}^d \sum_{t=1}^f \sum_{l=1}^L x_{ij} \right] = \frac{1}{\pi} \left( \frac{m^n}{\Gamma(m)\Omega^n} \right) \theta^n.
\]

Using Eq. (12) into Eq. (2), one obtains \( P_i^J(f) \) and thus the system FEP \( P_i^J(F) \).

### 3.2 FEP of protocol II

In this protocol, only portions of the entire frame will be retransmitted. Suppose that the frame is divided into \( L \) segments, where the \( i \)th segment has a length of \( N_i \) and a transmission attempt of \( f_i(f_i \geq 1) \), \((i = 1,2,\ldots,L)\). \( N_i \) and \( f_i \) can be determined by the destination according to the scheme in Ref. [4]. Then, the \( j \)th received bit in the \( i \)th segment will be

\[
y_{ij} = \sum_{t=1}^f \lambda_{ij} x_{ij} + \sum_{j=1}^d n_{ij}
\]

Assume that the \( i \)th segment contributes \( d_i \) bits to the overall output weight \( d \) of the entire frame [3]. Then the PEP of \( P_i^J(f) \) is given by

\[
P_i^J(d) = E \left[ \sum_{j=1}^d \sum_{t=1}^f \sum_{l=1}^L x_{ij} \right] = \frac{1}{\pi} \left( \frac{2m^n}{\Gamma(m)\Omega^n} \right) \theta^n.
\]

By inputting Eq. (17) into Eq. (16), the upper bound will be

\[
P_i^J(d) \leq E \left[ \sum_{j=1}^d \sum_{t=1}^f \sum_{l=1}^L x_{ij} \right]^2 = \frac{1}{\pi} \left( \frac{2m^n}{\Gamma(m)\Omega^n} \right) \theta^n.
\]

It seems impossible to get a closed-form expression of Eq. (14), thus the corresponding upper bound will be considered. Since

\[
\sum_{i=1}^L \sum_{j=1}^d f_i \sum_{l=1}^d \left( \sum_{j=1}^d \sum_{l=1}^L x_{ij} \right)^2 \leq \epsilon \sum_{i=1}^L \sum_{j=1}^d \left( \sum_{j=1}^d \sum_{l=1}^L x_{ij} \right)^2
\]

with \( \epsilon = \max(f_i) \), Eq. (14) is bounded by

\[
P_i^J(d) \leq E \left[ \sum_{j=1}^d \sum_{t=1}^f \sum_{l=1}^L x_{ij} \right]^2
\]

It should be noted that when each segment is retransmitted with the same attempts, the upper bound in Eq. (16) will become exact. Using the approximation in Eq. (8), one has

\[
\left( \sum_{i=1}^L d_i \right)^2 = (1 + 2A(f_i - 1)) \left( \sum_{j=1}^d \sum_{l=1}^L x_{ij} \right)^2
\]

By inputting Eq. (17) into Eq. (16), the upper bound will be

\[
P_i^J(d) \leq E \left[ \sum_{j=1}^d \sum_{t=1}^f \sum_{l=1}^L x_{ij} \right] = \frac{1}{\pi} \left( \frac{2m^n}{\Gamma(m)\Omega^n} \right) \theta^n.
\]

with \( \lambda_i = (2E_s/f_{\text{SNR}})(1 + 2A(f_i - 1)) \), \((i = 1,2,\ldots,L)\). When the signal to noise ratio (SNR) is sufficiently high, Eq. (18) becomes

\[
P_i^J(d) \leq E \left[ \sum_{j=1}^d \sum_{t=1}^f \sum_{l=1}^L x_{ij} \right]^2 = \frac{1}{2\pi} \left( \frac{2m^n}{\Gamma(m)\Omega^n} \right) \theta^n.
\]

with the help of Ref. [7]. Thus the upper bound is derived as

\[
P_i^J(d) \leq E \left[ \sum_{j=1}^d \sum_{t=1}^f \sum_{l=1}^L x_{ij} \right]^2 = \frac{1}{2\pi} \left( \frac{2m^n}{\Gamma(m)\Omega^n} \right) \theta^n.
\]
FEP can be obtained. It is noted from Eqs. (12) and (20) that the diversity orders in these two protocols are both related with $m$ and the transmission attempts.

### 3.3 Throughput

Based on the derived error performance, it is ready now to give the corresponding throughput for different protocols. By ignoring the reduced throughput induced by the use of CRC, the normalized throughput of two protocols can be both obtained as

$$\eta = \frac{R(1 - P_0(F))}{T_r}$$  \hspace{1cm} (21)

where $T_r$ is the average number of transmissions per frame.

In protocol I, it is given by [3]

$$T_r = 1 + \sum_{f=1}^{F-1} N_f P_0(f)$$  \hspace{1cm} (22)

In protocol II, suppose that $N_f$ bits out of $N$ are selected to be retransmitted at the $f$th transmission attempt, then $T_r$ in Eq. (21) will be

$$T_r = 1 + \sum_{f=4}^{F-1} \frac{N_f P_0(f)}{N}$$  \hspace{1cm} (23)

### 4 Simulation results

This section provides simulations to verify the analysis results in Sect. 3. It can be observed from Fig. 1 that the pdf approximations become less precise when $m$ decreases. To have a strong persuasion, the authors investigate the performance under relatively bad channel condition, i.e., Rayleigh fading ($m = 1$). The 1/2-code rate (7,5) (octal) convolutional code in Ref. [10] is utilized. Assume that the frame size is $N=128$ bit and the average channel gain is $\Omega = 1$.

At different values of $F$, Fig. 2 gives both the simulated and analytical FEPs of protocol I, where the agreement is reasonably good. It is noted that the diversity order increases with the increasing number of maximum retransmission number.

The FEP performance of protocol II is given in Fig. 3, where a frame is divided into two segments of the same length. The authors have the entire frame at the first transmission and a randomly selected segment at subsequent transmissions transmitted. As expected, different $F$ also implies different diversity order. Since less information is received at the destination, the FEP is worse than that of protocol I. However, it is soon found that a higher throughput is exhibited in protocol II because fewer messages are required to be retransmitted.

Fig. 4 shows the normalized throughput of protocols I and II, where they both converge to the code rate 1/2 when the SNR is high enough. It is observed that the throughput increases with $F$, but with a reduced gain. As expected, protocol II embraces a better throughput performance compared with protocol I. Thus, protocol II will be a more preferable choice in practice.
5 Conclusions

In this article, the authors develop the performance of HARQ over Nakagami-\(m\) fading channels, where two different retransmission protocols, protocols I and II, with coherent EGC are adopted. Specifically, when the destination requests retransmission, the same frame will be repeated in protocol I, while several selected parts of the frame are repeated in protocol II. As mentioned above, the recently proposed RBHARQ protocol is accommodated in protocol II. By approximating the Nakagami product to Gamma sum, the FEP and throughput are studied under different transmission attempts. Simulations are provided to verify the reliability of the analysis, and the results also well confirm the advantages of protocol II upon protocol I in terms of throughput.

Cooperative ARQ is recently proposed to exploit the space diversity introduced by cooperative nodes [11]. The corresponding performance is expected to be formulated by utilizing the analytical methods presented in this article, which is under investigation.

Acknowledgements

This work was supported by the National Basic Research Program of China (2007CB310604, 2009CB320401), and the National Natural Science Foundation of China (N60772108, 60702048).

References


(Editor: WANG Xu-ying)